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# TECHNICAL NOTE

## D-324

THE DESIGN OF FEEDBACK CONTROL SYSTEMS CONTAINING  
A SATURATION TYPE NONLINEARITY

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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A SATURATION TYPE NONLINEARITY

By Stanley F. Schmidt and Eleanor V. Harper

SUMMARY

This report is an extension of NASA TN D-20. The extension includes a derivation of the optimum response for a step input for plant transfer functions which have an unstable pole and further data on plants with a single zero in the left half of the  $s$  plane. The calculated data are presented tabulated in normalized form.

Optimum control systems are considered. The optimum system is defined as one which keeps the error as small as possible regardless of the input, under the constraint that the input to the plant (or controlled system) is limited. Intuitive arguments show that in the case where only the error can be sensed directly, the optimum system is obtained from the optimum relay or on-off solution. References to known solutions are presented. For the case when the system is of the sampled-data type, arguments are presented which indicate the optimum sampled-data system may be extremely difficult if not impossible to realize practically except for very simple plant transfer functions.

Two examples of aircraft attitude autopilots are presented, one for a statically stable and the other for a statically unstable airframe. The rate of change of elevator motion is assumed limited for these examples. It is shown that by use of nonlinear design techniques described in NASA TN D-20 one can obtain near optimum response for step inputs and reasonable response to sine wave inputs for either case. Also, the nonlinear design prevents inputs from driving the system unstable for either case.

INTRODUCTION

Saturation (or limiting) in a feedback control system can be analyzed and compensated for (if required) by an appropriate application of the root-locus and switch-time methods described in reference 1. The principles of these two methods are reviewed here for the reader's convenience.

To apply the root-locus method one draws the loci of the roots of the characteristic equation as a function of the equivalent<sup>1</sup> gain of the saturating device (equivalent limiter gain). One then views the closed-loop poles as moving on these loci as a function of the input magnitude. If the poles stay in well-damped regions of the  $s$  plane for all expected values of equivalent gain, one is assured that saturation causes no instability of the closed-loop system. Conversely, if the closed-loop poles move to lightly damped or unstable regions of the  $s$  plane for low values of equivalent limiter gain (large input magnitudes), the system response will be oscillatory or unstable when it is subjected to very large transients. In order to obtain qualitative ideas of how to compensate for the instability one studies how various changes in feedback parameters shift the zeros of the root-locus diagram. Shifts which are suitable are those that allow the equivalent limiter gain to decrease while the closed-loop poles stay in well-damped regions of the  $s$  plane. The feedback parameters (or zeros) are made to shift with one or more functions which are similar to the limiter input (such as the error).

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The switch-time method is based on the fact that the existence of saturation allows one to define an optimum response. The optimum response uses the saturated variable in such a manner as to restore the error and its derivatives to zero in a minimum time. It has been shown (ref. 1) that the first reversal time (after the application of a step) of the saturated variable can be used to obtain a quantitative measure for comparing the particular system to the optimum. If the actual first reversal time is longer than the optimum, then overshoot exists; conversely, if it is smaller, the actual response time is longer than the optimum. By designing nonlinear functions to make the actual first reversal time equal to the optimum for large step inputs, one obtains a design which has a stable nonoscillatory response for all input magnitudes.

As can be seen, the application of the switch-time method requires a knowledge of the optimum response for the particular "plant"<sup>2</sup> under consideration. A method for obtaining this type data was described in reference 1 where it was shown that if the plant transfer function has a zero in the left half of the  $s$  plane, then a considerable reduction in the response time to a step input can be achieved. The solution derived, however, requires some of the variables of the plant to change in a certain manner while the error and its derivatives remain zero.

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<sup>1</sup>Describing function analysis shows that saturation (or limiting) can be treated as an equivalent gain whose value decreases as the input magnitude increases (see, e.g., ref. 2).

<sup>2</sup>The word "plant" refers to the controlled system. The plant transfer function could, for example, be the mathematical relationship between the output of an amplifier (which saturates for large inputs) and a motor shaft position in a servo position controller.

One limitation of reference 1 was that no systems were derived to demonstrate that the above solution, obtained for a plant with a zero in its transfer function, was practically realizable.

A second limitation was that the limiter input equations were not derived for the optimum system. The optimum system is defined here as one which maintains the error as small as possible regardless of the input or initial condition under the constraint of finite limits on the plant input.

A third limitation was that no optimum curves or examples were derived which considered the problem of controlling an unstable plant (i.e., the transfer function has a pole in the right half of the  $s$  plane). This problem has considerable practical importance in certain autopilot problems where the aircraft is statically unstable in certain areas of its flight envelope.

It is the purpose of this investigation to remove these limitations. In particular, the following subjects are investigated and some practical solutions studied or suggested.

1. The optimum control system for plants with a saturation-type nonlinearity. The plant in this case is stable and has no zeros.

2. The influence of a zero in the plant transfer function on the optimum control system. The examples considered are for a type 1<sup>3</sup> second-order plant and a type 2 third-order plant.

3. The influence of an unstable pole in the plant transfer function on the optimum system for a type 1 second-order plant.

4. The determination of the optimum response of plant transfer functions of the type encountered in pitch-rate and pitch-attitude autopilots where the control surface rate is limited. This amounts to considering plant transfer functions of the form

$$(a) \quad \frac{K(\tau_1 s + 1)}{s \left( \frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1 \right)}$$

$$(b) \quad \frac{K(\tau_1 s + 1)}{s^2 \left( \frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1 \right)}$$

<sup>3</sup>Type in this usage was defined in reference 1 as the number of poles at the origin in the  $s$  plane. A type 1 second-order plant (with one zero) has a transfer function,  $G(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$ ; similarly a type 2 third-

order plant (with zero) has a transfer function,  $G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$ .

where  $\xi$  can take on all positive values and

$$(c) \quad \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)} \quad (d) \quad \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)}$$

The (c) and (d) cases are for statically unstable airframes.

5. Practical examples of an attitude autopilot. For this, two airframes are considered to demonstrate the difference in control systems for stable and unstable airframes.

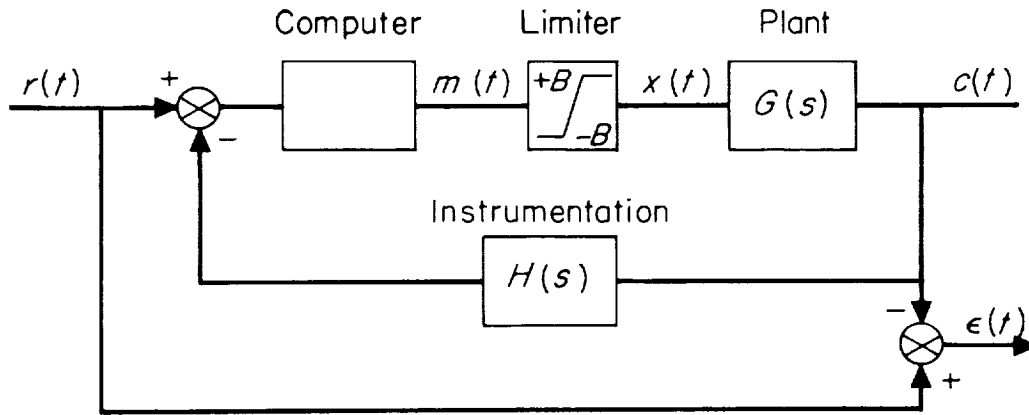
#### NOTATION

B	limit level
c(t)	controlled quantity
C(s)	Laplace transform of controlled quantity
D(s)	controller transfer function
D(z)	digital controller transfer function
e	2.7183
E(s)	Laplace transform of error
G(s)	plant transfer function
$h_p$	density altitude, ft
H(s)	feedback transfer function
K	gain
$L[f(t)]$	Laplace transform of $f(t)$
m(t)	controller output
M	Mach number
r(t)	input reference quantity
$R_0$	input step magnitude
R(s)	Laplace transform of reference quantity

s	Laplace operator
t	time, sec
T	sampling period, sec
$T_1$	first reversal time, sec
$T_m$	minimum response time, sec
$u(t)$	unit step function
$U(s)$	Laplace transform of unit step function
$x(t)$	limiter output (saturated variable)
$X(s)$	Laplace transform of saturated variable
z	$e^{-sT} = Z$ transform operator
$\delta$	control surface deflection, radians
$e(t)$	error function, $r(t) - c(t)$
$\zeta$	damping ratio
$\theta$	pitch angle, radians
$\tau$	time constant, sec
$\omega_n$	natural frequency, radians/sec

#### DESCRIPTION OF THE PROBLEM

The problem is best described with the help of the block diagram of sketch (a). The computer in this diagram will be assumed capable of providing any linear or nonlinear relationship desired between its input and output. The output of the computer,  $m(t)$ , will be assumed to be continuous if an analog computer is used or a staircase signal (a zero-order hold circuit on the output of a digital computer) if a digital computer is used.



Sketch (a)

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The output of the computer,  $m(t)$ , is fed to a limiter. The output of the limiter,  $x(t)$ , obeys the following equations

for

$$|m(t)| \leq B \quad (1)$$

$$x(t) = m(t)$$

for

$$|m(t)| > B \quad (2)$$

$$x(t) = B \operatorname{sgn}[m(t)]$$

The plant is assumed to be linear and representable by a transfer function,  $G(s)$ . The plant transfer functions in table I are considered here. The first seven cases were also treated in reference 1.

The feedback transfer function is  $H(s)$ . In general, one desires the error,  $e(t) = r(t) - c(t)$ , to be as small as possible for all the inputs,  $r(t)$ , the system receives. As will be shown later, this implies that  $H(s) = 1$  and all compensation (linear or nonlinear) be accomplished by the computer.

The general problem is how to design the computer operations in order to maintain the error as small as possible under the constraint that  $|x(t)| \leq B$ . This, as will be shown, generally implies very complicated computer equations for the plant transfer functions for which solutions have been found. At the present time, the solution is not known for most



plants whose characteristic equation is of third or higher order. Thus, in this investigation we are principally interested in finding approximate solutions.

The problem will be investigated in the following order:

1. Determination of the optimum switching times for the plants given in table I. This will be done in the section entitled "Optimum Responses."

2. Derivation (based on intuitive argument) of the optimum system. This subject is covered in the section entitled "Optimum Systems." Reference data to known previous work as well as new work which includes simple plants with an unstable pole or stable zero are contained in this section.

3. Approximate optimum systems for

a. A pitch attitude autopilot for a stable airframe.

b. A pitch attitude autopilot for a statically unstable airframe.

These systems are derived in the section entitled "Approximate Optimum Systems."

## OPTIMUM RESPONSES

A general theorem which was partially proved in reference 1 is as follows: If a Laplace transformable function of time,  $f(t)$ , is truncated (i.e.,  $f_T(t) = 0$  for  $a > t > b$  where  $f_T(t)$  is the truncated time signal), then the Laplace transform of  $f_T(t)$ ,  $L[f_T(t)]$ , is an entire function.

This theorem can be used to advantage in deriving the optimum response if further information regarding the shape of  $x(t)$ , the bounded variable, is attainable. Consider sketch (a) for  $r(t) = R_0 u(t)$  and zero initial conditions. Then

$$L[\epsilon(t)] = E(s) = \frac{R_0}{s} - X(s) G(s) \quad (3)$$

A necessary but not a sufficient condition if we are to attain the optimum is that  $E(s)$  must be an entire function because  $\epsilon(t) = 0$  for  $t < 0$  and we desire  $\epsilon(t) = 0$  for  $t > T_m$ . In other words, the desired  $\epsilon(t)$  is a truncated time signal. In order to meet the sufficiency condition one must know the optimum shape of  $x(t)$  in general terms of unknown reversal times. One then derives  $X(s)$  and determines the relationship between these unknown reversal times and  $R_0$  by forcing equation (3) to be an entire function.

The general shape of  $x(t)$  for the minimum response time can be found by either inspection, or by use of a theorem proved by Bellman, Glicksberg, and Gross (ref. 3) for the conditions for which it applies. The theorem applied to this problem states that, in order to have a minimum response time to an input step,  $x(t)$  should be at its maximum value plus or minus throughout the response, the maximum number of reversals being equal to  $n-1$  where  $n$  is the order of the system. The proof, however, is only for  $G(s)$  having real, distinct, and negative roots. It is shown later that if one confines his interest to plants having only poles in the left half plane or at  $s = 0$  and is interested only in responses to step inputs, the theorem gives a sufficient number of reversals of  $x(t)$  in order to restore the error and its derivatives to zero in a finite time, that is, to allow  $E(s)$  given by equation (3) to be an entire function. As will be demonstrated, the response time obtained by using this number of reversals may not be the minimum for plants with lightly damped complex poles. It will also be shown here that if zeros exist in  $G(s)$  then one does not want the bounded variable,  $x(t)$ , to be at its maximum throughout the entire response, that is, until the time when all the states of the system are restored to zero or a constant.

The steps involved in obtaining the optimum responses are summarized below.

(1) Determine  $x(t)$  in terms of the unknown switch times for the optimum response. This function will be the limit level times a sum of delayed steps for  $G(s)$  having only poles. The delay times will be written as undetermined coefficients.

(2) Determine  $X(s)$  from  $x(t)$  with the switching times as parameters.

(3) Use the fact that the error transform must be an entire function to obtain algebraic or transcendental equations relating  $R_0$  and the unknown switching times and exponential decays (if any) of  $x(t)$ .

(4) Solve the equations of step (3) to obtain the optimum first reversal time and the minimum response time as functions of the input step magnitude,  $R_0$ .

In reference 1 these steps were followed in deriving the optimum responses for the first 7 cases of table I. The optimum responses for cases 8 to 12 will be derived here.

#### A Type 1 Second-Order Unstable Plant

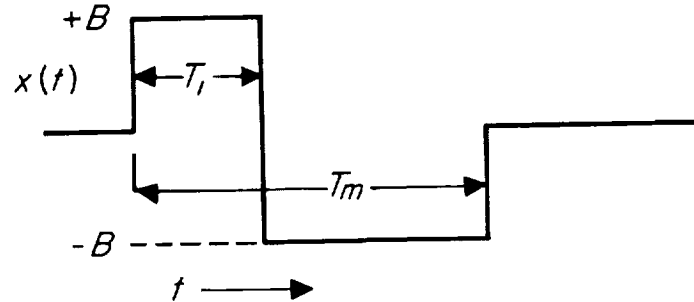
The plant transfer function for this example (case 8, table I) is  $G(s) = K/s(\tau_s - 1)$ . Arguments similar to those presented in reference 1

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show that for  $x(t) = R_0 u(t)$  where  $u(t)$  is the unit step function,  $x(t)$  must be of the form shown in sketch (b).



Sketch (b)

The Laplace transform of  $x(t)$  is

$$X(s) = B \frac{1 - 2e^{-T_1 s} + e^{-T_m s}}{s} \quad (4)$$

The Laplace transform of the error is (see eq. (3))

$$E(s) = \frac{R_0}{s} - BK \frac{1 - 2e^{-T_1 s} + e^{-T_m s}}{s^2(s\tau - 1)} \quad (5)$$

The relationships which must be satisfied in order that equation (5) be an entire function are

$$1 - 2e^{-T_1/\tau} + e^{-T_m/\tau} = 0 \quad (6)$$

$$\frac{R_0}{BK} = T_m - 2T_1 \quad (7)$$

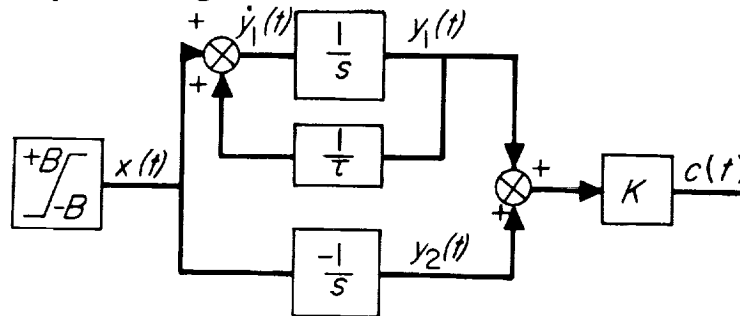
from which one can find

$$\frac{R_0}{BK} = \tau \ln \frac{1}{2e^{-T_1/\tau} - 1} - 2T_1 \quad (8)$$

$$\frac{R_0}{BK} = T_m - 2\tau \ln \frac{2}{1 + e^{-T_m/\tau}} \quad (9)$$

Equations (8) and (9) are normalized by dividing by  $\tau$  and the curves showing  $T_1/\tau$  and  $T_m/\tau$  versus  $R_0/BK\tau$  are presented in figure 1.

As can be seen from figure 1(a), for the unstable plant the optimum first reversal time approaches a constant for  $R_0$  approaching infinity. Since this condition will exist for all the unstable plants considered here, the reason for its existence will be derived. The derivation is accomplished by expanding the plant transfer function by means of partial fractions and by defining state variables as shown in sketch (c).



Sketch (c)

As can be seen from sketch (c), the transfer function,  $Y_1(s)/X(s)$ , contains the unstable mode (pole in the right half plane). Thus, if  $x(t)$  is a step function and  $y_1(0) = 0$ , then  $y_1(t)$  is a growing exponential.

From sketch (c) it is seen that the input to the integrator,  $\dot{y}_1(t)$ , is  $\dot{y}_1(t) = x(t) + (1/\tau)y_1(t)$ . If  $y_1(0) = 0$  and  $x(t) = Bu(t)$ , then the input to the integrator grows with time. If control is to be maintained of  $y_1(t)$  by  $x(t)$ , then  $x(t)$  must be able to reverse the sign of  $\dot{y}_1(t)$ . This is impossible if  $y_1(t) > \tau B$  when  $x(t)$  is limited to a magnitude of  $B$ . Thus if we use the optimum motion of  $x(t)$  shown in sketch (b),  $T_1$  must never exceed a certain value. This value is, in general, (from eq. (8))  $T_1 = \tau \ln 2$ .

This simple derivation is easily shown for the second-order example by phase plane analysis. It amounts to the fact that  $y_1(t)$  must be held within certain limits. If  $y_1(t)$  is ever allowed out of these limits, then  $c(t)$  grows indefinitely, regardless of  $x(t)$ . This is the principal difference between stable and unstable plants; that is, with an unstable plant the phase space must be limited if one is to control the plant with a limited input.

#### A Type 1 Third-Order Stable Plant (With Zero)

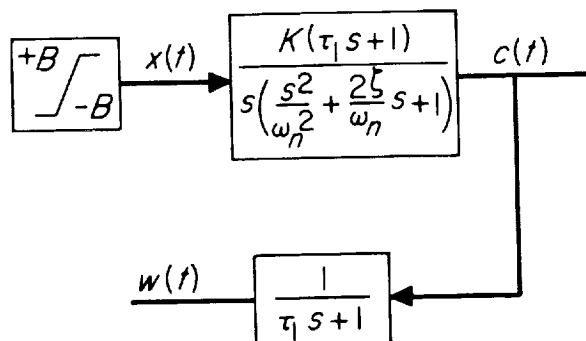
For this example (case 9 of table I)

$$G(s) = \frac{K(\tau_1 s + 1)}{s[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$$

The problem is to determine the optimum motion of  $x(t)$ . There are several possibilities. One scheme is to use a pole to cancel the zero,

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as illustrated in sketch (d). If this solution is used,  $w(t)$  will be a zero overshoot response for a step input to the system. The output,  $c(t)$ , will overshoot, however. The amount of overshoot can be extremely large if  $\tau_1$  is large. It therefore would seem desirable to find a solution which has zero overshoot.

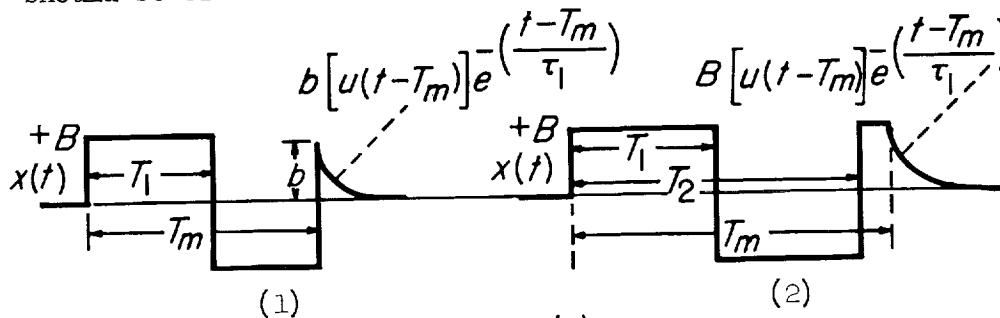


Sketch (d)

It was shown in reference 1 for a simple plant transfer function that if  $x(t)$  is a bang-bang motion followed by a tailing off exponential of time constant  $\tau_1$ , then  $c(t)$  has no overshoot. With this solution all of the states of the plant are not zero when the error and its derivatives are zero and there are practical questions as to how such a system can be implemented. The questions will be resolved later in this report.

For the more complicated plant considered here one questions whether a solution of the type desired can be found by simply taking more switching times for the bang-bang part of  $x(t)$ . This actually turns out to be the case. The solution which is derived here was actually used in an example in reference 4 and thus one has experimental evidence of its existence.

From these considerations one concludes that the optimum motion of  $x(t)$  should be of the form shown in sketch (e). A partial proof that these



Sketch (e)

motions are optimum is obtained if we are successful in finding solutions to the equations which force  $E(s)$  to be an entire function; these solutions must have a shorter minimum response time than the one obtained by the scheme shown in sketch (d).

If  $x(t)$  is of the form shown in sketch (e 1), the equations which force  $E(s)$  to be an entire function are

$$\frac{R_0 \omega_n}{BK} = 2\omega_n T_1 - \omega_n T_m + \frac{b}{B} \omega_n \tau_1 \quad (10)$$

$$(1 - 2e^{\alpha \omega_n T_1} + e^{\alpha \omega_n T_m})(1 - \alpha \omega_n \tau_1) - \frac{b}{B} \alpha \omega_n \tau_1 e^{\alpha \omega_n T_m} = 0 \quad (11)$$

$$(1 - 2e^{\bar{\alpha} \omega_n T_1} + e^{\bar{\alpha} \omega_n T_m})(1 - \bar{\alpha} \omega_n \tau_1) - \frac{b}{B} \bar{\alpha} \omega_n \tau_1 e^{\bar{\alpha} \omega_n T_m} = 0 \quad (12)$$

where

$$\alpha = \zeta + \sqrt{\zeta^2 - 1}$$

$$\bar{\alpha} = \zeta - \sqrt{\zeta^2 - 1}$$

If  $|b/B|$  from this solution is greater than unity, one must use the  $x(t)$  motion given in sketch (e 2). If this motion is used, the equations which force  $E(s)$  to be an entire function are

$$\frac{R_0 \omega_n}{BK} = 2\omega_n T_1 - 2\omega_n T_2 + \omega_n T_m + \omega_n \tau_1 \quad (13)$$

$$(1 - 2e^{\alpha \omega_n T_1} + 2e^{\alpha \omega_n T_2} - e^{\alpha \omega_n T_m})(1 - \alpha \omega_n \tau_1) - \alpha \omega_n \tau_1 e^{\alpha \omega_n T_m} = 0 \quad (14)$$

$$(1 - 2e^{\bar{\alpha} \omega_n T_1} + 2e^{\bar{\alpha} \omega_n T_2} - e^{\bar{\alpha} \omega_n T_m})(1 - \bar{\alpha} \omega_n \tau_1) - \bar{\alpha} \omega_n \tau_1 e^{\bar{\alpha} \omega_n T_m} = 0 \quad (15)$$

where

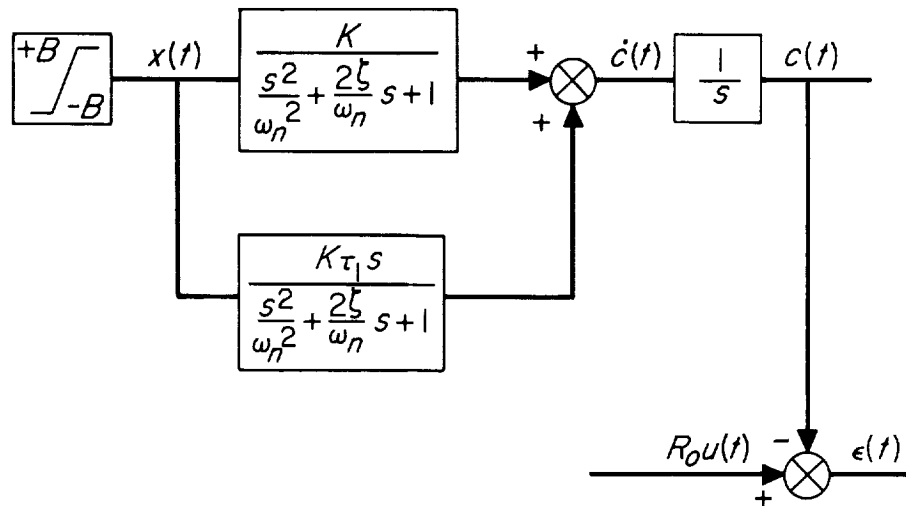
$$\alpha = \zeta + \sqrt{\zeta^2 - 1}$$

$$\bar{\alpha} = \zeta - \sqrt{\zeta^2 - 1}$$



As was the case for the type 1 third-order example considered in reference 1, these equations cannot be solved analytically. It is therefore necessary to resort to an iterative procedure and use a digital computer. The digital computer was programmed to solve equations (10), (11), and (12). If the solution obtained has a value of  $|b/B| > 1$ , this solution is discarded and equations (13), (14), and (15) are used.

As it turns out, for  $\zeta > 1$ , a sensible solution to these equations always exists. For  $\zeta < 1$ , however, a sensible solution does not exist for all values of the variables. One can give a physical argument which substantiates this fact. Consider the block diagram of the plant shown in sketch (f).



Sketch (f)

In order for  $c(t)$  to reach  $R_0$  (for  $R_0+$ ) in the minimum time,  $\dot{c}(t)$  should have the highest value permissible in the interval. The quantity,  $x(t)$ , during a large portion of this interval is equal to the positive limit level,  $B$ . The motion of  $\dot{c}(t)$ , until  $T_1$ , is seen from sketch (f) to be the sum of two components. Since  $x(t) = Bu(t)$  for  $t < T_1$ ,  $\dot{c}(t)$  can be expressed as (for  $t < T_1$ )

$$\dot{c}(t) = KB \left[ \left( \begin{array}{c} \text{unit step response} \\ \text{of a second-order} \\ \text{system} \end{array} \right) + \tau_1 \left( \begin{array}{c} \text{unit impulse response} \\ \text{of a second-order} \\ \text{system} \end{array} \right) \right] \quad (16)$$

The unit step and impulse response of a second-order system for various values of  $\zeta$  is given in figure 2.

As can be seen from figure 2 (or proved mathematically) for  $\zeta > 1$ , the impulse response is positive for all time. Since the unit step response is positive (for  $\zeta > 0$ ), we can say that  $\dot{c}(t)$  will be positive until  $T_1$  if  $\zeta > 1$ , regardless of the value of  $\tau_1$ .

If  $\zeta < 1$ , however, the unit impulse is negative part of the time. This means that  $\dot{c}(t)$ , given by equation (16), can reverse sign for certain values of  $t$ ,  $\zeta$ , and  $\tau_1$ . For small values of  $T_1$ , however,  $\dot{c}(t)$  will be positive so a sensible solution using the  $x(t)$  motion given in sketch (e) exists. These facts generally imply that the  $x(t)$  motion chosen is non-optimum for all values of the variables. Rather than attempting to find other motions of  $x(t)$ , we have chosen to present the data where it is a sensible solution. Since the possible size step input or transient to most physical systems is bounded and since sensible solutions exist for a certain range of the step input (starting at zero), this restriction of scope is perhaps not as serious as it could be.

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The data summarizing the switch times,  $b/B$ , and input step magnitude for various values of  $\zeta$  and  $\tau_1$  are presented in table II. It is believed that the range of the parameters chosen,  $\zeta$  and  $\tau_1$ , should be sufficient for a large number of control systems.

A typical plot of  $\omega_n T_1$  and  $\omega_n T_m$  versus  $R_0 \omega_n / KB$  for  $\zeta = 0.5$  which shows the effects of  $\omega_n \tau_1$  is presented in figure 3. As can be seen from this figure a sensible solution exists for all the plotted values of  $\omega_n \tau_1$  except  $\omega_n \tau_1 = 16$ . Here, the solution does not exist for  $R_0 \omega_n / KB \gtrsim 18$ .

As can be seen also, the response time is much shorter for large values of  $\omega_n \tau_1$  than for  $\omega_n \tau_1 = 0$ . Thus a large benefit in speed of response of the system is possible in cases where the plant has a zero close to the origin in comparison with other poles.

#### A Type 1 Third-Order Unstable Plant (With Zero)

The plant transfer function for this example (case 10, table I) is

$$G(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$$

Arguments identical to the ones given for the previous example indicate the  $x(t)$  motion should be one of the forms shown in sketch (e). Note also that the results of the second-order unstable plant presented earlier indicate  $T_1$  should approach a finite limit as  $T_m$  and  $R_0$  approach infinity.

If the motion of  $x(t)$  shown in sketch (e) is assumed, the equations which must be satisfied in order that  $E(s)$  be an entire function are

$$\frac{R_0}{BK\tau_2} = \frac{T_m}{\tau_2} - \frac{2T_1}{\tau_2} - \frac{b}{B} \frac{\tau_1}{\tau_2} \quad (17)$$

$$\left(1 - 2e^{T_1/\tau_2} + e^{T_m/\tau_2}\right) \left(1 - \frac{\tau_1}{\tau_2}\right) - \frac{b}{B} \frac{\tau_1}{\tau_2} e^{T_m/\tau_2} = 0 \quad (18)$$

$$\left[1 - 2e^{-\frac{T_1}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)} + e^{-\frac{T_m}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)}\right] \left[1 + \frac{\tau_1}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)\right] + \frac{b}{B} \frac{\tau_1}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right) e^{-\frac{T_m}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)} = 0 \quad (19)$$

Equations (17) to (19) are normalized in a manner such that solutions for  $R_0/BK\tau_2$ ,  $T_m/\tau_2$ ,  $T_1/\tau_2$ , and  $b/B$  are desired for various values of the ratios  $\tau_1/\tau_2$  and  $\tau_2/\tau_3$ . The ratio  $\tau_1/\tau_2$  can be expressed as

$$\frac{\tau_1}{\tau_2} = \frac{\text{distance of stable pole from the origin of the } s \text{ plane}}{\text{distance of stable zero from the origin of the } s \text{ plane}}$$

similarly

$$\frac{\tau_2}{\tau_3} = \frac{\text{distance of unstable pole from the origin of the } s \text{ plane}}{\text{distance of stable pole from the origin of the } s \text{ plane}}$$

Expressing the ratios in this form gives an understanding of their meaning in terms of a pole-zero plot of the plant transfer function. One can also see that if  $\tau_1/\tau_2 \gg 1$  that one might eliminate  $\tau_2$  as an approximation and use data for a simpler plant. Also note that for  $\tau_1/\tau_2 = 1$  the pole and zero cancel and the data for the second-order unstable plant presented earlier is the solution.

If the solution of these equations gives a value of  $|b/B|$  greater than unity, one must discard the answer and derive the solution using the motion of  $x(t)$  given in sketch (e 2). If  $x(t)$  is of this form, the equations which must be satisfied in order that  $E(s)$  be an entire function are

$$\frac{R_0}{BK\tau_2} = \frac{2T_2}{\tau_2} - \frac{2T_1}{\tau_2} - \frac{T_m}{\tau_2} - \frac{\tau_1}{\tau_2} \quad (20)$$

$$\left(1 - 2e^{-\frac{T_1}{\tau_2}} + 2e^{-\frac{T_2}{\tau_2}} - e^{-\frac{T_m}{\tau_2}}\right)\left(1 - \frac{\tau_1}{\tau_2}\right) - \frac{\tau_1}{\tau_2} e^{-\frac{T_m}{\tau_2}} = 0 \quad (21)$$

$$\left[1 - 2e^{-\frac{T_1}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)} + 2e^{-\frac{T_2}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)} - e^{-\frac{T_m}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)}\right]\left(1 + \frac{\tau_1}{\tau_3}\right) + \frac{\tau_1}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right) e^{-\frac{T_m}{\tau_2} \left(\frac{\tau_2}{\tau_3}\right)} = 0 \quad (22)$$

As was the case for the previous example, these two sets of equations, (17) to (19) and (20) to (22) must be solved by an iterative procedure. However, this example is somewhat simpler than the previous one since one can derive solutions which are valid for  $T_m \rightarrow \infty$ . In equation (19) for example, if  $T_m \rightarrow \infty$  we find

$$T_1 = \tau_3 \ln 2 \quad (23)$$

Similarly, one can divide equation (18) by  $e^{T_m/\tau_2}$  and then let  $T_m \rightarrow \infty$  and find

$$\frac{b}{B} = \frac{\tau_2}{\tau_1} - 1 \quad (24)$$

Equation (23) shows that  $T_1$  approaches the same limit for  $T_m \rightarrow \infty$  as for the second-order example considered earlier. For the reasons given earlier, this is the maximum value of  $T_1$ .

Equation (24) shows that if  $\tau_2/\tau_1 \leq 2$  (or  $\tau_1/\tau_2 \geq 0.5$ ) then as  $T_m \rightarrow \infty$ ,  $|b/B| \leq 1$ . As it turns out,  $b/B$  is very small for  $T_m$  small and the value given by equation (24) is the maximum regardless of the value of  $T_m$ .

Equations (17) to (19) were solved by means of a digital computer for the quantities  $R_0/BK\tau_2$ ,  $T_1/\tau_2$ ,  $T_m/\tau_2$ , and  $b/B$  for various values of the ratios  $\tau_1/\tau_2$  and  $\tau_2/\tau_3$ . Only one value of  $\tau_2/\tau_1 > 2$  (i.e.,  $\tau_1/\tau_2 = 0$ ) was computed; therefore equations (20) to (22) were needed for this case only.

The results of the computations are presented in table III. A typical plot of the tabulated results showing the effects of the ratio,  $\tau_1/\tau_2$ , for one value of the ratio,  $\tau_2/\tau_3$ , is presented in figure 4. As can be seen from figure 4(b), a zero in the left half plane much closer

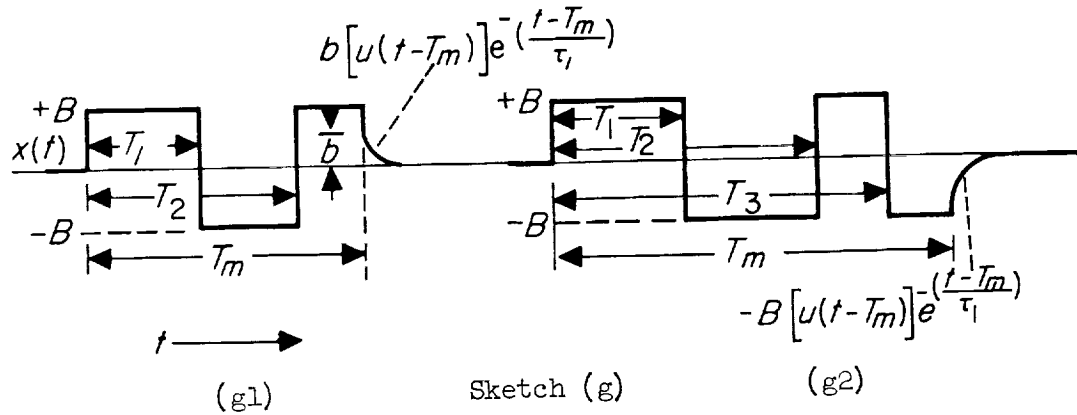
to the origin than a pole (i.e.,  $\tau_1/\tau_2 \gg 1$ ) provides a considerably smaller response time than if  $\tau_1/\tau_2 = 0$ . Thus, as was the case for the previous example, a left half plane zero of the plant can be used to advantage if a control system can be designed which provides the bang-bang followed by a tailing off exponential motion required of the saturated variable,  $x(t)$ .

### A Type 2 Fourth-Order Plant (With Zero)

The plant transfer function for this example (case 11, table I) is

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]} \quad (25)$$

Arguments similar to the ones given previously show the optimum motion of  $x(t)$  should be one of the two forms shown in sketch (g).



Note that these two motions are identical to sketch (e) except that an additional switch time has been added to the bang-bang part of the motion.

If the  $x(t)$  motion is as shown in sketch (g 1), the equations which must be satisfied in order that  $E(s)$  be an entire function are

$$2\omega_n T_1 - 2\omega_n T_2 + \omega_n T_m + \frac{b}{B} \omega_n \tau_1 = 0 \quad (26)$$

$$\left(1 - 2e^{\alpha\omega_n T_1} + 2e^{\alpha\omega_n T_2} - e^{\alpha\omega_n T_m}\right) \left(1 - \alpha\omega_n \tau_1\right) - \frac{b}{B} \alpha\omega_n \tau_1 e^{\alpha\omega_n T_m} = 0 \quad (27)$$

$$\left(1 - 2e^{\bar{\alpha}\omega_n T_1} + 2e^{\bar{\alpha}\omega_n T_2} - e^{\bar{\alpha}\omega_n T_m}\right) \left(1 - \bar{\alpha}\omega_n \tau_1\right) - \frac{b}{B} \bar{\alpha}\omega_n \tau_1 e^{\bar{\alpha}\omega_n T_m} = 0 \quad (28)$$

$$\begin{aligned} \frac{2R_0\omega_n^2}{BK} &= 2\omega_n \tau_1 (2\omega_n T_1 - 2\omega_n T_2 + \omega_n T_m) - 2(\omega_n T_1)^2 + 2(\omega_n T_2)^2 \\ &\quad - (\omega_n T_m)^2 - 2 \cdot \frac{b}{B} (\omega_n \tau_1) (\omega_n T_m) \end{aligned} \quad (29)$$

where

$$\left. \begin{aligned} \alpha &= \zeta + \sqrt{\zeta^2 - 1} \\ \bar{\alpha} &= \zeta - \sqrt{\zeta^2 - 1} \end{aligned} \right\} \quad (30)$$

If the motion of  $x(t)$  is as shown in sketch (g 2), the equations which must be satisfied in order that  $E(s)$  be an entire function are

$$2\omega_n T_1 - 2\omega_n T_2 + 2\omega_n T_3 - \omega_n T_m - \omega_n \tau_1 = 0 \quad (31)$$

$$\left(1 - 2e^{\alpha\omega_n T_1} + 2e^{\alpha\omega_n T_2} - 2e^{\alpha\omega_n T_3} + e^{\alpha\omega_n T_m}\right) \left(1 - \alpha\omega_n \tau_1\right) + \alpha\omega_n \tau_1 e^{\alpha\omega_n T_m} = 0 \quad (32)$$

$$\left(1 - 2e^{\bar{\alpha}\omega_n T_1} + 2e^{\bar{\alpha}\omega_n T_2} - 2e^{\bar{\alpha}\omega_n T_3} + e^{\bar{\alpha}\omega_n T_m}\right) \left(1 - \bar{\alpha}\omega_n \tau_1\right) + \bar{\alpha}\omega_n \tau_1 e^{\bar{\alpha}\omega_n T_m} = 0 \quad (33)$$

$$\begin{aligned} \frac{2R_0\omega_n^2}{BK} &= 2(\omega_n \tau_1)^2 - 2(\omega_n T_1)^2 + 2(\omega_n T_2)^2 - 2(\omega_n T_3)^2 \\ &\quad + (\omega_n T_m)^2 + 2(\omega_n \tau_1) (\omega_n T_m) \end{aligned} \quad (34)$$

Equations (26), (27), and (28) are solved for  $\omega_n T_1$ ,  $\omega_n T_2$ , and  $b/B$  for assumed values of  $\omega_n T_m$ ,  $\xi$ , and  $\omega_n \tau_1$ . If the value of  $|b/B|$  obtained is less than unity, equation (29) is used to obtain  $R\omega_n^2/BK$ . If the value of  $|b/B| > 1$ , the solution is discarded and equations (31), (32), and (33) are solved for  $\omega_n T_1$ ,  $\omega_n T_2$ , and  $\omega_n T_3$  for the same values of  $\omega_n T_m$ ,  $\xi$ , and  $\omega_n \tau_1$ . Equation (34) is used for the correct value of  $R\omega_n^2/BK$ . The equations were solved by an iterative method on a digital computer. The results are presented in table IV. A typical plot of the results is given in figure 5. As was the case for the previous examples, it can be seen from figure 5(b) that the minimum response time is reduced as  $\tau_1$  is increased.

As was the case for the type 1 third-order stable plant (with zero) considered earlier, a sensible solution to the equations does not exist for  $\xi < 1$  for certain values of  $\omega_n \tau_1$  and  $\omega_n T_m$ . The reason for this is similar to the one presented earlier except, in this case, the output acceleration reverses sign for certain values of  $\omega_n \tau_1$  and  $\omega_n T_m$  in the interval  $t < T_1$  rather than the output velocity as for the previous case. The  $x(t)$  motion of sketch (g) is nonoptimum for these ranges of parameters. Rather than attempting to find the optimum motion of  $x(t)$  for all ranges of parameters we have chosen to present the data where it is a sensible solution.

#### A Type 2 Fourth-Order Unstable Plant (With Zero)

The plant transfer function for this example (case 12, table I) is

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)} \quad (35)$$

The optimum motion of  $x(t)$  is one of the two forms shown in sketch (g). As was the case for the previous unstable example  $T_1$  must approach a limit as  $T_m \rightarrow \infty$ .

If the motion of  $x(t)$  is as shown in sketch (g 1), the equations which must be satisfied in order that  $E(s)$  be entire are

$$2 \frac{T_1}{\tau_2} - 2 \frac{T_2}{\tau_2} + \frac{T_m}{\tau_2} + \frac{b}{B} \frac{\tau_1}{\tau_2} = 0 \quad (36)$$

$$\left(1 - 2e^{\frac{T_1}{\tau_2}} + 2e^{\frac{T_2}{\tau_2}} - e^{\frac{T_m}{\tau_2}}\right) \left(1 - \frac{\tau_1}{\tau_2}\right) - \frac{b}{B} \frac{\tau_1}{\tau_2} e^{\frac{T_m}{\tau_2}} = 0 \quad (37)$$

$$\left[ 1 - 2e^{-\frac{T_1}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} + 2e^{-\frac{T_2}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} - e^{-\frac{T_m}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} \right] \left[ 1 + \frac{\tau_1}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right) \right] + \frac{b}{B} \left( \frac{\tau_1}{\tau_2} \right) \left( \frac{\tau_2}{\tau_3} \right) e^{-\frac{T_m}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} = 0 \quad (38)$$

$$\frac{2R_0}{BK\tau_2^2} = -2 \frac{\tau_1}{\tau_2} \left( 2 \frac{T_1}{\tau_2} - 2 \frac{T_2}{\tau_2} + \frac{T_m}{\tau_2} \right) + 2 \left( \frac{T_1}{\tau_2} \right)^2 - 2 \left( \frac{T_2}{\tau_2} \right)^2 + \left( \frac{T_m}{\tau_2} \right)^2 + 2 \frac{b}{B} \left( \frac{\tau_1}{\tau_2} \right) \left( \frac{T_m}{\tau_2} \right) \quad (39)$$

If the motion of  $x(t)$  is as shown in sketch (g 2), the equations which must be satisfied in order that  $E(s)$  be entire are

$$2 \frac{T_1}{\tau_2} - 2 \frac{T_2}{\tau_2} + 2 \frac{T_3}{\tau_2} - \frac{T_m}{\tau_2} - \frac{\tau_1}{\tau_2} = 0 \quad (40)$$

$$\left( 1 - 2e^{\frac{T_1}{\tau_2}} + 2e^{\frac{T_2}{\tau_2}} - 2e^{\frac{T_3}{\tau_2}} + e^{\frac{T_m}{\tau_2}} \right) \left( 1 - \frac{\tau_1}{\tau_2} \right) + \frac{\tau_1}{\tau_2} e^{\frac{T_m}{\tau_2}} = 0 \quad (41)$$

$$\left[ 1 - 2e^{-\frac{T_1}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} + 2e^{-\frac{T_2}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} - 2e^{-\frac{T_3}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} + e^{-\frac{T_m}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right)} \right] \left[ 1 + \left( \frac{\tau_1}{\tau_2} \right) \left( \frac{\tau_2}{\tau_3} \right) \right] - \frac{\tau_1}{\tau_2} \left( \frac{\tau_2}{\tau_3} \right) e^{-\frac{T_m}{\tau_2} \frac{\tau_2}{\tau_3}} = 0 \quad (42)$$

$$\frac{2R_0}{BK\tau_2^2} = -\frac{2\tau_1^2}{\tau_2^2} + 2 \left( \frac{T_1}{\tau_2} \right)^2 - 2 \left( \frac{T_2}{\tau_2} \right)^2 + 2 \left( \frac{T_3}{\tau_2} \right)^2 - \left( \frac{T_m}{\tau_2} \right)^2 - 2 \left( \frac{\tau_1}{\tau_2} \right) \left( \frac{T_m}{\tau_2} \right) \quad (43)$$

The solution of these equations is by an iterative procedure so similar to previous examples that no further explanation is considered necessary. The solutions are presented in table V for the same values of  $\tau_2/\tau_3$  and  $\tau_1/\tau_2$  as were given for the type 1 third-order plant presented earlier.



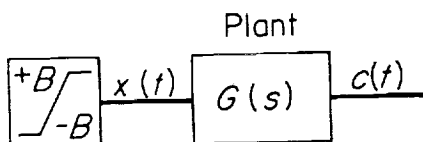
Typical plots of the data are shown in figure 6. As can be seen from figure 6(a),  $T_1/\tau_2$  approaches the limit  $T_1/\tau_2 = (\tau_3/\tau_2)\ln 2$  as  $R_0/BK\tau_2^2 \rightarrow \infty$ . This is the same limit value as for the previous unstable case. As can be seen from figure 6(b), the minimum response time for a particular input decreases as  $\tau_1$  increases which has been true for all the plants considered.

#### A Method for Determining Approximate Switch Time Data

The data presented in this report and reference 1 cover a large number of plant transfer functions. The examples given here (in later sections) and those of reference 1 serve to show the switch time method has fairly general application if one designs to (a) force the first reversal time to be optimum and (b) force the zeros to shift in the  $s$  plane in such a manner that as the equivalent gain of the limiter reduces, the closed-loop poles remain in reasonably damped regions. In the examples considered it is only necessary to change the zero positions as a function of the error. Other examples may require further complications or additions; however, it is believed the general concepts are still applicable.

One of the limitations in applying the concept is the limited optimum switching time data available. The plant may be nonlinear or have a transfer function for which data are not available. It is the purpose of this section to show an approximate method for obtaining these data. The method is well suited to analog computer computation or tests on the actual hardware.

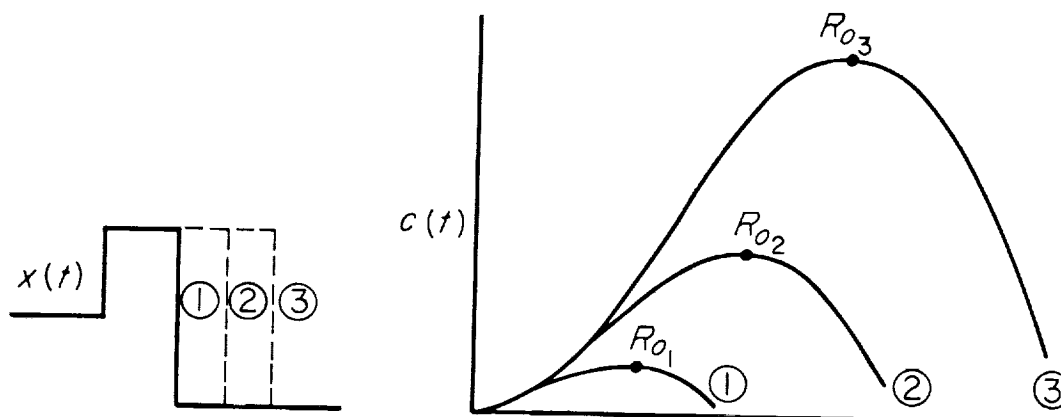
Consider the system of sketch (h). It is desired to know the optimum



Sketch (h)

first reversal time,  $T_1$ , as a function of input step magnitude,  $R_0$ . It has generally been shown that the optimum motion of  $x(t)$  is to remain at the maximum in one direction (+B) until  $T_1$  and then reverses to (-B), the maximum in the opposite direction. As the order of the system is increased, further switchings take place at later times.

The  $c(t)$  motions for several  $x(t)$  motions which are maximum one way and reverse at  $T_1$  will generally be as shown in sketch (i).



Sketch (i)

To obtain an approximate curve one could plot the peak value of  $c(t)$ , ( $R_{01}$ ,  $R_{02}$ ,  $R_{03}$ ), versus the switching times (1, 2, 3). If the plant is simulated on an analog computer or if actual hardware is tested such plots are easily accomplished since the  $x(t)$  motion of sketch (i) can be readily generated by electronic devices such as interval timers. This simple scheme was originally suggested in reference 4. It can be seen to be exact for second-order plants since the output velocity is zero at  $c(t)$  maximum and returning  $x(t)$  to zero at this time (or using a tailing off exponential for plants with a zero in the left half plane) will cause no further change in  $c(t)$ .

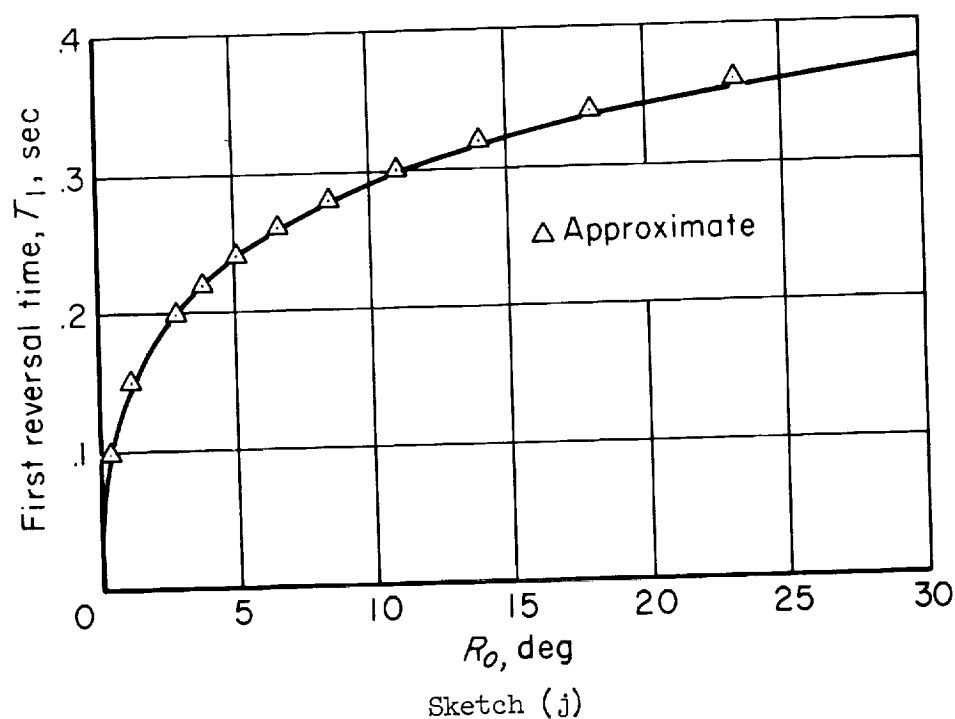
For higher order plants one questions the accuracy of such a simple technique. To obtain an idea of the relative accuracy the relationships between  $R_0$  and  $T_1$  used in this scheme can be compared to the exact relationship. This has been done for type 3 third-order and type 4 fourth-order plants. The results are

<u>Plant transfer function</u>	<u>Exact</u>	<u>Approximate</u>
$\frac{K}{s^3}$	$T_1^3 = \frac{R_0}{2KB}$	$T_1^3 = \frac{R_0}{1.91 KB}$
$\frac{K}{s^4}$	$T_1^4 = \frac{R_0(0.1766)}{KB}$	$T_1^4 = \frac{R_0(0.2107)}{KB}$

As can be seen, the error is about 5 percent for the third-order plant and 15 percent for the fourth-order plant. If some of the roots of the plant denominator are in the left half plane, it is believed the error

will be less than that given by the above cases. The reason for this belief is that some of the output derivatives are bounded (by the fact that  $x(t)$  is bounded) when there are poles in the left half plane. The result is that the ensuing motions required to force all derivatives of  $c(t)$  to be zero have less effect than in the cases shown above. To obtain an idea of the relative accuracy for unstable plants approximate data are compared in sketch (j) with exact data (from table V) for a plant transfer function

$$G(s) = \frac{3.125(s + 1)}{s^2(0.183s + 1)(0.683s - 1)}$$



As can be seen, the approximate data, obtained from an analog computer simulation, are in close agreement with the exact curve.

One point should be mentioned with regard to the use of approximate first reversal time data. That is, the approximate  $T_1$  is slightly larger than the exact  $T_1$ . The result is that one will obtain a design which has some overshoot. This overshoot is usually readily removed by slight adjustments in the nonlinear function.

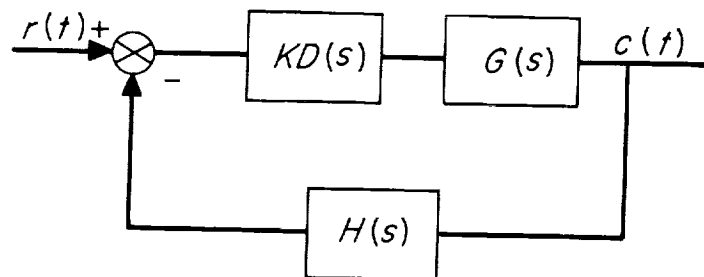
## OPTIMUM SYSTEMS

It is the purpose of this section to consider the problem of designing an optimum feedback control system. This type system<sup>4</sup> will be defined as one which maintains the error as small as possible under the constraint that the plant input variable is bounded. Unfortunately, very little is known about such systems at the present time. If the system is continuous of the type shown in sketch (a), and it is subjected to initial conditions only ( $r(t) = 0$ ), it is generally accepted that the relay or bang-bang system is the optimum.<sup>5</sup> If the system is of the sampled-data type ( $m(t)$  of sketch (a) is a staircase signal), little is known about the design of the optimum system.

In this section the discussion will be restricted to the plant transfer functions given in table I.

## The Optimum Continuous Linear System

Prior to discussing cases where saturation is present, it is pertinent to discuss what the previous definition of an optimum system means in terms of a linear system. To have the optimum (i.e., zero error to any input) it is quite obvious that the closed-loop transfer function relating  $C(s)/R(s)$  of sketch (k) must be unity.



Sketch (k)

<sup>4</sup>Another definition of optimum is made when the input is contaminated with noise. The optimum system is then defined as one which minimizes the quantity  $H_d r(t) - c(t)$ . The input (without noise) is  $r(t)$ ;  $c(t)$  is the output; and  $H_d$  is the mathematical representation of the desired response which may or may not be physically realizable (see, e.g., ref. 5).

<sup>5</sup>Optimum in the sense that the error is restored to zero in the minimum time.

Since the closed-loop transfer function must be unity then

$$\frac{C(s)}{R(s)} = 1 = \frac{KD(s)G(s)}{1 + KD(s)G(s)H(s)} \quad (44)$$

Thus, one obtains the following two conditions if equation (44) is to be satisfied:

$$H(s) = 1 \quad (45)$$

$$K = \infty \quad (46)$$

A third condition is obtained from stability considerations:

$$D(s)G(s) = \frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \dots} \quad (47)$$

The roots of the numerator of equation (47) must have real parts which are negative and the order of the numerator can only be 1 lower than the order of the denominator. The following arguments based on root-locus rules verify this.

a. For  $K \rightarrow \infty$  all poles of the root locus equation move to zeros; therefore, all zeros must be in the left half of the  $s$  plane or the system will be unstable. The roots of the numerator must, therefore, have negative real parts.

b. If the order of the numerator is 3 (or higher) less than the denominator, some of the closed-loop poles for  $K \rightarrow \infty$  are in the right half plane; therefore, the system will be unstable.

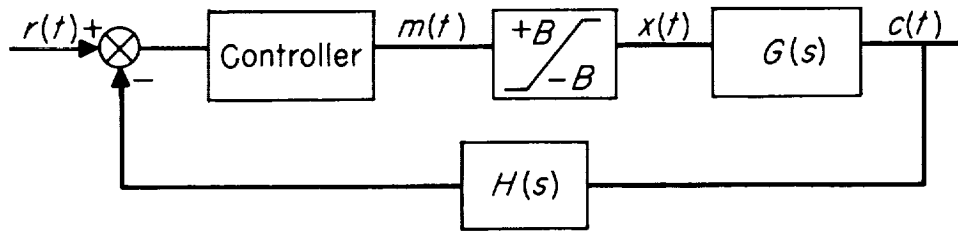
c. If the order of the numerator is 2 lower than the denominator, two of the poles move to infinity (as  $K \rightarrow \infty$ ) with asymptotes which have a finite real part. This real part can be negative; however, since it is finite, the impulse response time is finite. Therefore, the closed-loop transfer function cannot be unity.

d. If the order of the numerator is 1 less than the denominator, one pole moves to infinity (as  $K \rightarrow \infty$ ) along the negative real axis. With  $K = \infty$  this provides a closed-loop transfer function of first order with a time constant equal to zero which is essentially a transfer function of unity.

The important point gained from this consideration of linear systems is that if  $G(s)$  has  $n$  poles and no zeros,  $n-1$  derivatives of the error must be available in order to design the optimum linear system.<sup>6</sup>

### The Optimum Continuous Control System With Saturation

Consider the system shown in sketch (1).



Sketch (1)

If the controller's characteristics are linearized about zero input, it can be seen that the same conditions given for the optimum linear system must be true for this system; that is,

$$H(s) = 1$$

$$\frac{M(s)}{E(s)}_{K \rightarrow \infty} = K(1 + a_1 s + a_2 s^2 + a_3 s^3 + \dots)$$

The roots of  $s$  of the equation determined by setting

$$\frac{M(s)}{E(s)} = 0$$

must be in the left half of the  $s$  plane. Furthermore, since  $x(t)$  is bounded and  $X(s)G(s) = C(s)$ , certain output derivatives or sums of output derivatives are bounded. For example if

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<sup>6</sup>Practically speaking, the optimum linear system cannot exist, since it is impossible to differentiate. Any practical differentiating circuits also introduce poles which, when included in the system, cause the system to show an instability at some finite value of gain,  $K$ . If the poles introduced by the differentiating circuits are sufficiently far from the origin in the  $s$  plane compared to the plant poles and to the zeros introduced by the differentiating circuits, then one can obtain a system design which approximates the optimum (with a finite gain) over a low-frequency band width.

$$G(s) = \frac{K}{s(\tau_2 s + 1)} \quad (48)$$

then (see sketch (1))

$$|x| = \left| \frac{\tau_2 \ddot{c} + \dot{c}}{K} \right| \leq B \quad (49)$$

If these same bounds are placed on the input which for this example gives

$$\left| \frac{\tau_2 \ddot{r} + \dot{r}}{K} \right| \leq B \quad (50)$$

then the plant has the capability of following the input exactly. Thus, bounding  $x(t)$  forces one to put certain bounds on derivatives of the input (depending on  $G(s)$ ) if zero error is to be obtained.

The question therefore is what can be said about the optimum system when equation (50) is not satisfied. One answer could be that, regardless of the present error and its derivatives, we would like  $x(t)$  to take the optimum motion to reduce the error to zero in a minimum time.<sup>7</sup>

If this answer is correct, then the characteristics of the controller<sup>8</sup> would be (with  $H(s) = 1$ )

$$m(t) = Kf(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots) \quad (51)$$

where

$$K = \infty$$

The function of error and its derivatives,  $f(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots)$ , is the equation for the optimum switching surface in the phase space  $\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots$ .

A second answer to the question has been pursued by Hopkin and Wang (ref. 6). Their answer amounts to making the error zero at some predicted time in the future. For this answer  $m(t)$  becomes

$$m(t) = Kf(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots, \dot{r}, \ddot{r}, \dots) \quad (52)$$

where

$$K = \infty$$

<sup>7</sup>The input could be such that it was impossible for the error to ever be zero. For example, the plant given by equation (48) could not follow a constant acceleration input in a system of the type given in sketch (1), since the limit on  $x(t)$  forces the output velocity to be bounded.

<sup>8</sup>The input to the plant  $x(t)$  can (for  $K = \infty$ ) be written as  $B \operatorname{sgn} f(\epsilon, \dot{\epsilon}, \ddot{\epsilon}, \dots)$ . There is a perfect equivalence between considering the controller to be an infinite gain device and the more customary way of lumping the controller and limiter together and calling the combination an ideal relay.

The derivatives of  $r$  occurring in equation (52) are a result of prediction of the future error. For the second-order example in which they had run simulation tests their system worked considerably better than a linear function of  $e$  and  $\dot{e}$  alone; however, no data were presented for the system of equation (52) where nonlinear functions are required.

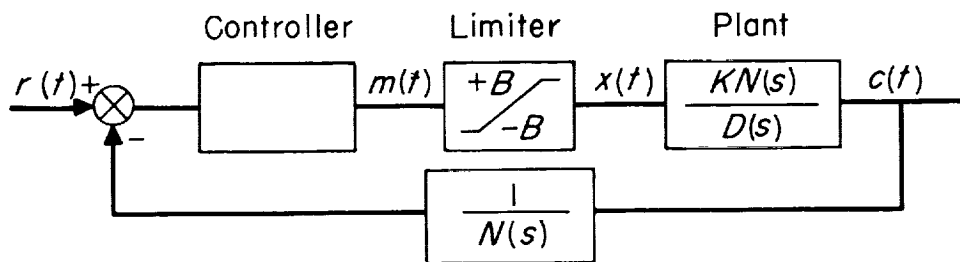
In many cases it is not possible to measure the input; that is, the error is sensed directly (for example radar or infrared seekers). For these cases the only choice for an optimum would be given by equation (51).

There is an apparent need for more work on the problem of the optimum system where saturation is present; however, we will assume the system of the type given by equation (51) is the optimum for many situations where the input is not sensed directly. If this assumption is valid, there are previous results available. Table VI lists previous work in which the optimum controllers of some of the plants (given by case number) can be found. The expressions are written out for the examples where this is possible. As can be seen from the table, to the authors' knowledge, only the simplest plants have been studied sufficiently by previous investigators to define the optimum switching surface. For these cases, a study of the reference for the particular plant involved allows one to determine the complexity of the control equation required if one desires the optimum system. The minimum response data of reference 1 and the previous section demonstrate what can be gained (compared to some linear controller) by the use of this usually complicated control equation.

A  
3  
9  
3

### The Influence of Plant Zeros on the Optimum System

It was shown in sections 3.5 and 3.8 of reference 1 and the previous section of this report that zeros in the plant transfer function make the minimum response time considerably lower than the value obtained when the zeros are neglected. A system which gives the relay, or bang-bang, solution can always be obtained with zeros in the plant transfer function (if they are in the left half plane) by the simple artifice of adding a lag network in the feedback path as was shown in sketch (d) and repeated in sketch (m) for the convenience of the reader.



Sketch (m)



The terms  $N(s)$  and  $D(s)$  represent the numerator and denominator of  $G(s)$ , respectively, written in polynomial form. The controller equation is then that given or referred to in table VI for the particular plant (if such data are available).

It has already been shown that this system is nonoptimum for step inputs and it is quite easy to show that this system is not the optimum for slow changing inputs. To show this, one linearizes the controller equations around zero input. For a very high gain controller then

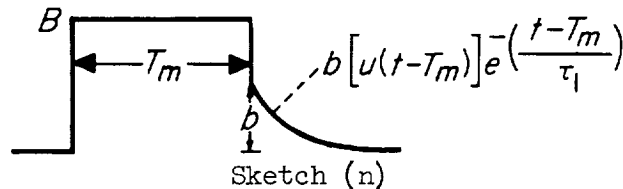
$$\frac{C(s)}{R(s)} \approx N(s) \quad (53)$$

or, the output "leads" the input. Two cases will be investigated in this section to show how one can use the knowledge gained thus far in deriving an optimum system.

A type 1 second-order plant.— For this case

$$G(s) = \frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$$

It was shown in chapter III of reference 1 that if one restricts his interest to plants where  $\tau_1/\tau_2 > 1/2$ , then the optimum motion for  $x(t)$  for step inputs is as shown in sketch (n). The error to step inputs is



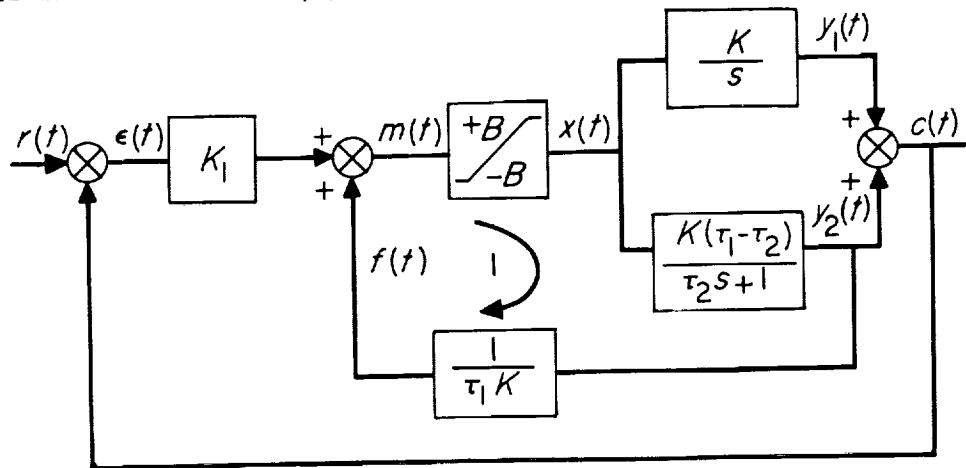
zero when  $t = T_m$ . Thus, the control system desired is one that holds  $x(t)$  on its limits until the error is zero and then causes the motion of  $x(t)$  to have an exponential decay of time constant,  $\tau_1$ .

The amplitude of the tailing off exponential,  $x(T_m) = b$ , is

$$b = B \left( 1 - e^{-\frac{T_m}{\tau_2}} \right) \left( 1 - \frac{\tau_2}{\tau_1} \right) \quad (54)$$

(See ref. 1, pp. 44-50 for the derivation of this equation.)

A control system which provides the optimum motion of  $x(t)$  of sketch (n) is shown in sketch (o).



Sketch (o)

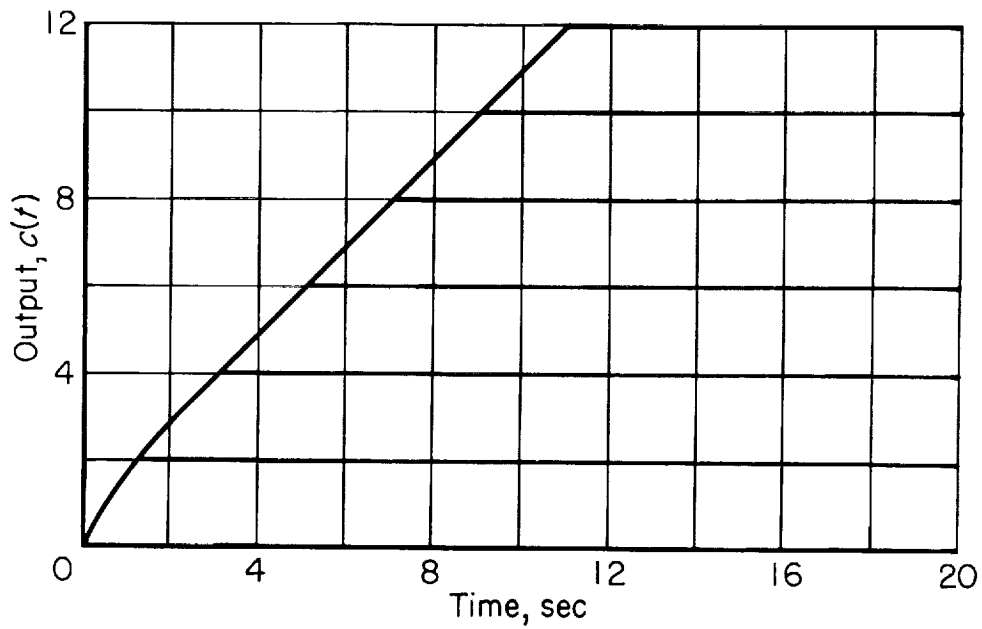
The quantity  $K_1$  of sketch (o) approaches infinity with the restriction that  $\epsilon K_1 = \infty$  if  $\epsilon$  has any value and  $\epsilon K_1 = 0$  if  $\epsilon = 0$ .

A study of sketch (o) shows that the main control of  $x(t)$  comes from the error signal  $e(t)$ . In other words, if the error has any value whatsoever,  $c(t)$  will be moving in such a direction to reduce the error as fast as possible; that is,  $\dot{c}(t)$  will be at its maximum. Only when the error is zero does the signal  $f(t)$  have any effect. The time constant of loop number 1 when  $x(t)$  is not at its limit can be readily calculated to be equal to  $\tau_1$ . The value of  $x(T_m)$  can be seen from sketch (o) to be equal to  $y_2(T_m)/\tau_1 K$ . Since  $y_2(0) = 0$ , the value of  $y_2(T_m)$  is readily calculated by the Laplace transform to be

$$y_2(T_m) = L^{-1} \left\{ \left( \frac{B}{s} \right) \left[ \frac{K(\tau_1 - \tau_2)}{(\tau_2 s + 1)} \right] \right\} \Big|_{T_m} = BK\tau_1 \left( 1 - \frac{\tau_2}{\tau_1} \right) \left( 1 - e^{-\frac{T_m}{\tau_2}} \right) \quad (55)$$

It can be seen that  $x(T_m) = y_2(T_m)/K\tau_1$  agrees with equation (54).

No system has been derived for the case of  $\tau_1/\tau_2 < 1/2$ ; however, since one reversal of  $x(t)$  is required before the error is zero, it is probable that a more complicated system is required than that shown in sketch (o). Sketch (p) shows some typical time histories of the output response,  $c(t)$ , for various step input magnitudes. For this example  $\tau_1 = 2$ ,  $K = 1$ ,  $\tau_2 = 1$ , and  $|B| = 1$ .

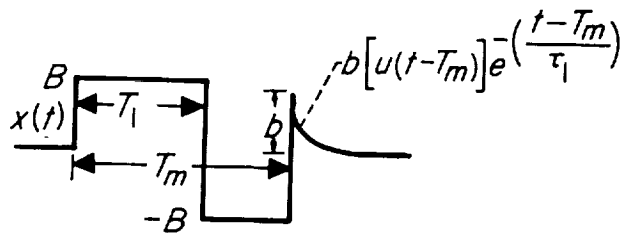


Sketch (p)

A type 2 third-order plant.— For this example

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$$

If one restricts his interest to examples where  $\tau_1/\tau_2 > 1/2$ , then the optimum motion of  $x(t)$  is as given in sketch (q).

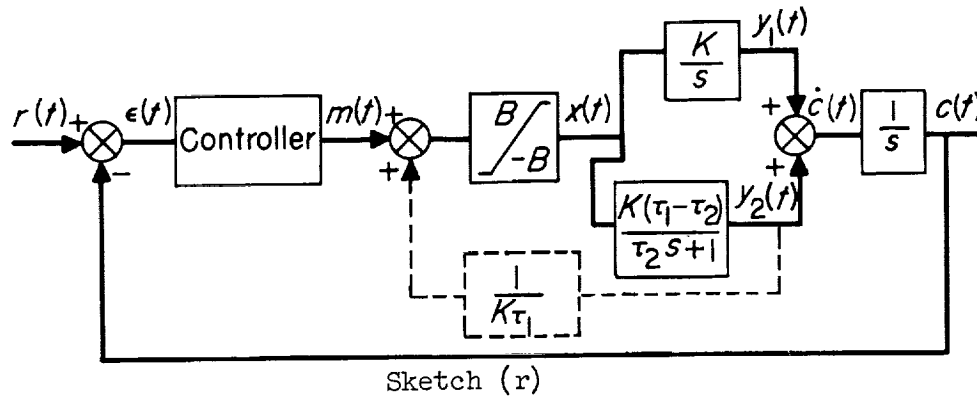


Sketch (q)

One can expand  $G(s)$  and define state variables as follows:

$$G(s) = \frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)} = \frac{1}{s} \left[ \frac{K}{s} + \frac{K(\tau_1 - \tau_2)}{\tau_2 s + 1} \right] = \frac{1}{s} \left[ \frac{Y_1(s)}{X(s)} + \frac{Y_2(s)}{X(s)} \right] \quad (56)$$

This system is shown in the following block diagram.



The dotted feedback in sketch (r) has been added here for the same reason as the previous example; that is, if we get the error and error rate to zero, this feedback allows  $y_1(t)$  and  $y_2(t)$  to change while  $\dot{c}(t)$  remains zero and  $c(t)$  remains a constant (or zero). The fact that the solution following  $T_m$  of sketch (r) will be correct can be seen by checking the time constant of the dotted loop as well as reviewing section 3.8 of reference 1.

The completion of the design requires that the controller equations be determined and realized. In order to overcome the dotted feedback the gain of the controller must be infinite. Consider step inputs only; the motion shown in sketch (q) shows that only one reversal need take place since, when  $t = T_m$ , both error and error rate are zero. This suggests that an equation which is a function of  $\epsilon$  and  $\dot{\epsilon}$  of form, for example

$$m(t) = [\epsilon + f(\dot{\epsilon})]K_1 \quad (57)$$

$$K_1 \rightarrow \infty$$

should be sufficient.

The nonlinear function,  $f(\dot{\epsilon})$ , given in equation (57) was calculated using the method presented in chapter II of reference 1 to make  $T_1$  equal to the optimum, and it was found that this scheme will not work since

$m(t)$  given by equation (57) does not reverse sign at  $T_1$  if  $x(t)$  is the shape of sketch (q).<sup>9</sup> The reason for this is best explained if the switching curve in the phase plane ( $\epsilon, \dot{\epsilon}$ ) is determined and an optimum trajectory for one input magnitude is shown. We know all the necessary information to compute both the trajectory and the optimum switching curve. For example

$$\epsilon(t) = R_0 u(t) - L^{-1} \left[ X(s) G(s) \right] \quad (58)$$

$X(s)$  is the Laplace transform of  $x(t)$  shown in sketch (q). The relationship between  $R_0$ ,  $T_1$ , and  $T_m$  is determined from the optimum curves of chapter III of reference 1. The error velocity is

$$\dot{\epsilon}(t) = R_0 \delta(t) - L^{-1} \left[ sX(s) G(s) \right] \quad (59)$$

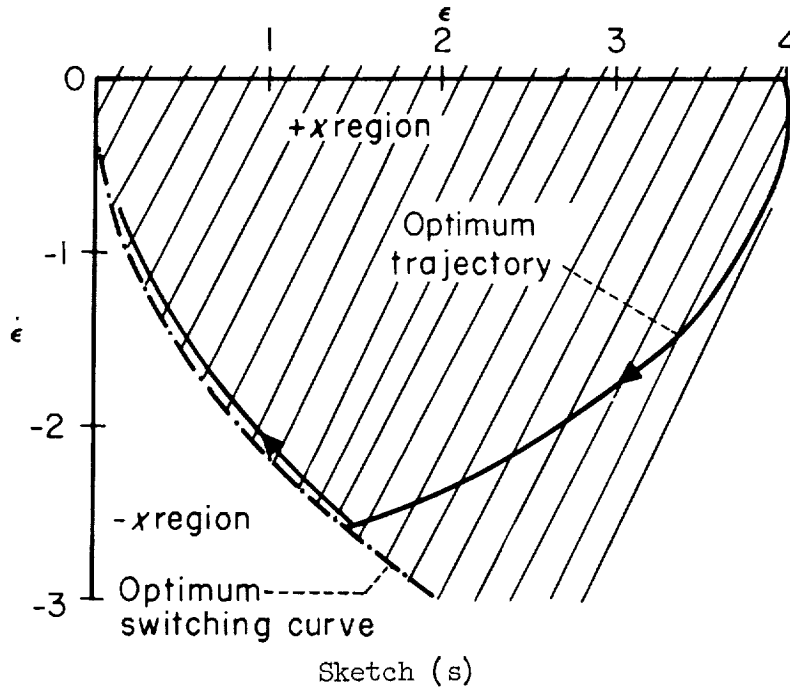
The  $\delta$  function may be omitted since it is only an impulse at the beginning of the trajectory. For any assumed value of  $R_0$ , one can therefore compute and plot  $\epsilon$  versus  $\dot{\epsilon}$ .

The optimum switching curve is found by plotting  $\epsilon(T_1)$  versus  $\dot{\epsilon}(T_1)$  as obtained from equations (58) and (59);  $R_0$  in equation (58) is obtained from the curves of chapter III (ref. 1) for each value of  $T_1$ .

The results of these computations for the example where  $B = 1$  and  $G(s) = (2s + 1)/s^2(s + 1)$  are shown in sketch (s). An examination of the optimum trajectory after it meets the optimum switching curve shows the reason why the switching function of equation (57) is unsatisfactory. The optimum trajectory stays in the  $+x$  region; therefore, a system which uses this scheme cannot be expected to be optimum. The simulation results showed the system to have a reasonably large first overshoot, but no undershoot to the response. The amount of overshoot was smaller than that which would be given by a system of the type shown in sketch (m); however, any overshoot at all means we do not have the optimum system.

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<sup>9</sup>A note should be added that this is not a limitation of the switch-time method. This method is based on the fact that we have a linear region and the switch-time method simply provides techniques for determining non-linear functions which provide near-optimum response for large step inputs. For the case at hand, we are attempting to use the switch-time method for determining an optimum switching surface in the phase space, since this is a third-order problem. We therefore cannot use the switch-time method directly and expect to have an optimum system.



What is really needed to obtain the optimum system is the equation for the switching surface in the phase space. This equation is not known at this time, so we shall, instead, design the system which will have optimum response to steps but nonoptimum response to other inputs. We must introduce a new variable in the system so that when the trajectory hits the switching surface,  $x$  will change sign. The particular system derived here will, in reality, be an approximate optimum system and should perhaps have been introduced in the next section. This was not done since a somewhat different design viewpoint will be introduced then.

It may be noted from sketch (s) that if a straight line is drawn from the origin to the desired switching point on the optimum switching curve, then the optimum trajectory will always be to the left of this line. This suggests that a switching equation of the type

$$m(t) = K_1[\epsilon + \dot{\epsilon}f(y_3)] \quad (60)$$

$$K_1 \rightarrow \infty$$

will be satisfactory if the variable  $y_3$  is "slow" compared with  $\dot{\epsilon}$ . There are probably a number of choices which could be made; however, for this example, the choice was that

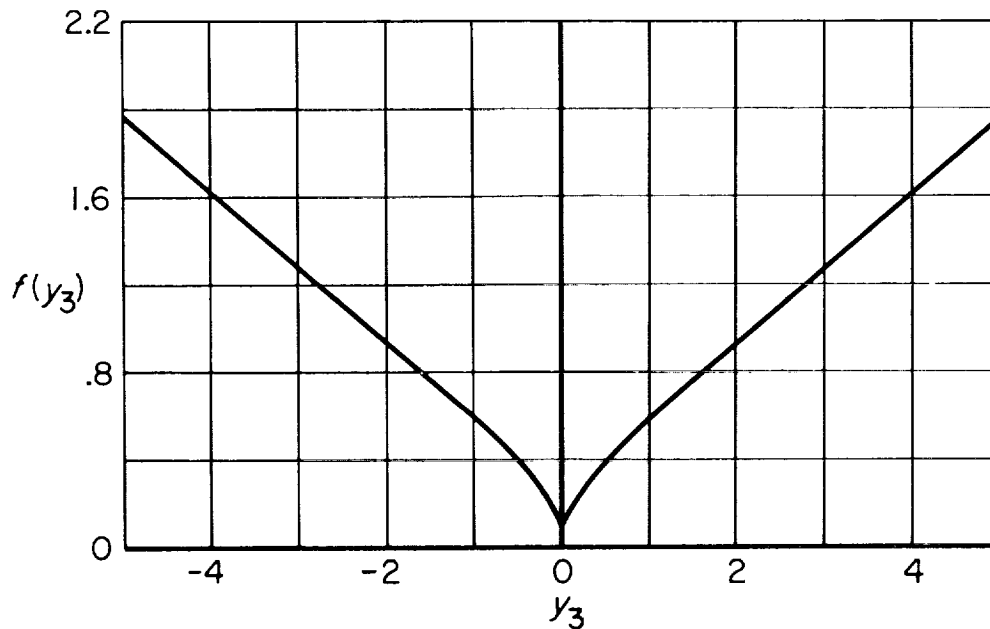
$$y_3(t) = y_1(t) - y_2(t) \quad (61)$$

For the particular values of  $\tau_1$  and  $\tau_2$  chosen for the example,

$$Y_3(s) = \frac{sC(s)}{2s + 1}$$

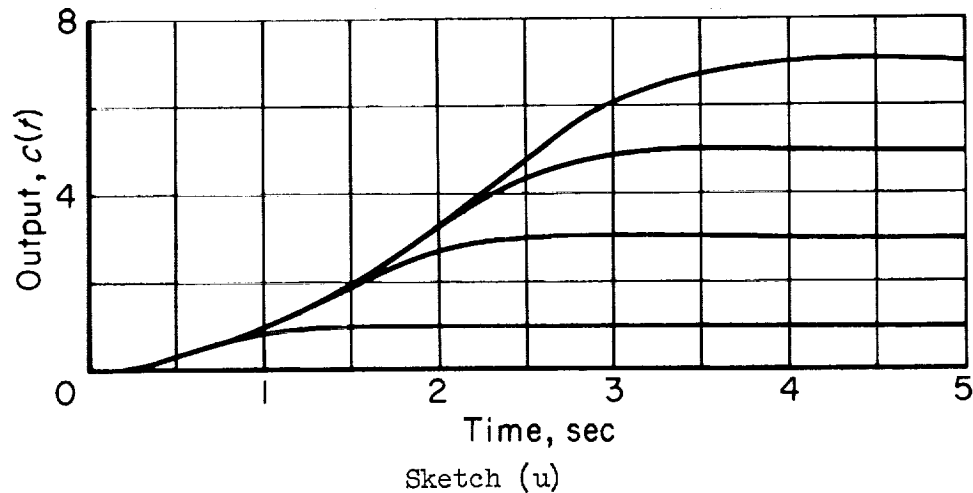
Thus  $y_3(t)$  is nothing more than the output velocity fed through a time lag. Since the choice was not based on phase space considerations, it, therefore, is not surprising that the results of tests conducted on the analog computer showed optimum response to steps and certain initial conditions but showed nonoptimum response to other initial conditions. In all cases where nonoptimum response was found, however, it was due to excessive damping rather than instabilities or low damping.

The function  $f(y_3)$  can be obtained for any example by forcing the first reversal time to be correct by the switch time method given in chapter II of reference 1. For the example case, the function  $f(y_3)$  is shown in sketch (t). The step responses obtained from the analog computer



Sketch (t)

are shown in sketch (u). As was mentioned, good response was obtained for a number of initial conditions; however, it was not the optimum for all conditions. The question remains then as to what the optimum controller is for this example. The need for further research for the case where the plant has zeros is certainly indicated.



### The Influence of an Unstable Pole on the Optimum System

As has been shown earlier, if the plant has an unstable pole, the optimum first reversal time of the saturated variable must approach a finite limit as  $R_0 \rightarrow \infty$ ; otherwise, control over the plant is impossible. To demonstrate what this implies in terms of the optimum system it is convenient to use a type 1 second-order plant. The optimum switching times for this example are presented in figure 1.

The plant transfer function is

$$G(s) = \frac{K}{s(\tau s - 1)}$$

Although one can obtain the optimum switching curve by either the switch-time method or phase plane for second-order problems (as was shown in ref. 1), it is believed the phase plane derivation is the simplest.

By use of the fact that  $x(t)$  is either  $\pm B$  one can derive the trajectory equations in the phase plane  $c(t)$ ,  $\dot{c}(t)$ . For initial conditions (or steps) one can substitute  $\epsilon$  for  $-c$  and  $\dot{\epsilon}$  for  $-\dot{c}$ . For this example

$$\tau \ddot{c}(t) - \dot{c}(t) = \pm KB \quad (62)$$



Let

$$\left. \begin{aligned} c(t) &= x & \dot{c}(t) &= y \\ \text{then} & & \ddot{c}(t) &= \frac{dy}{dt} = \frac{dy}{dt} \left( \frac{dt}{dx} \right) y = y \frac{dy}{dx} \end{aligned} \right\} \quad (63)$$

Substituting the relationships of equation (63) in (62) one obtains

$$\frac{dy}{dx} = \frac{1}{\tau} \left( \frac{y \pm KB}{y} \right) \quad (64)$$

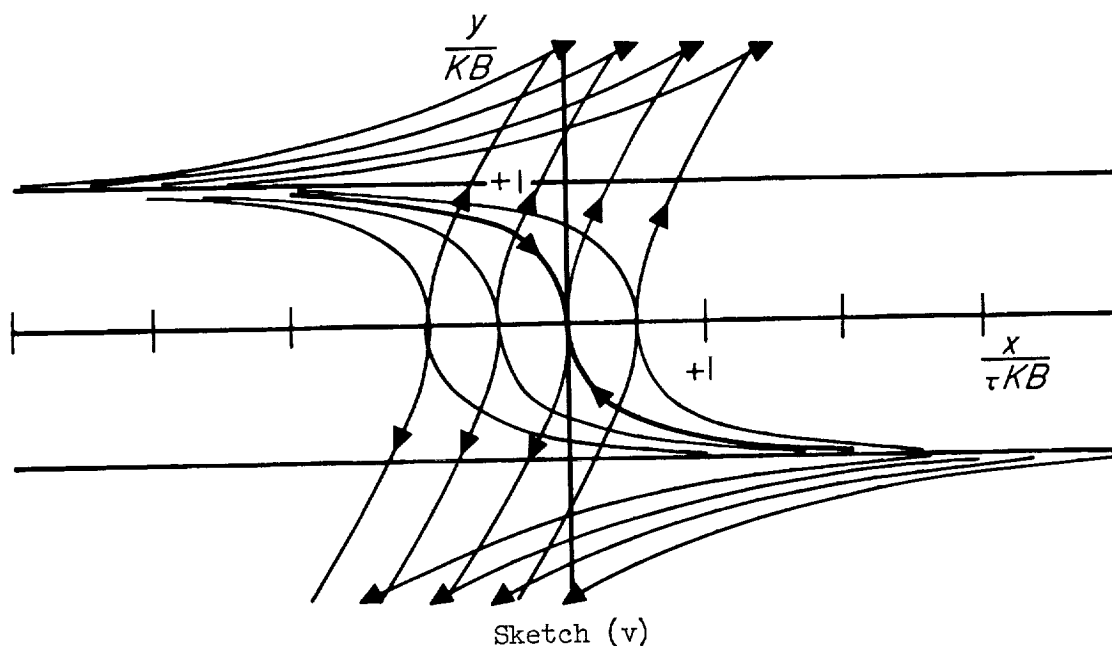
The solution of equation (64) is

$$\frac{x}{\tau} = y \mp KB \ln(y \pm KB) + C \quad (65)$$

If equation (65) is divided by  $KB$  one obtains

$$\frac{x}{\tau KB} = \frac{y}{KB} \mp \ln(\pm KB) \mp \ln \left( 1 + \frac{y}{\pm KB} \right) + C_1 \quad (66)$$

If  $C_1$  is chosen to be  $\pm \ln(\pm KB)$  then  $x$  will be zero when  $y$  is zero. With this choice of constant, equation (66) represents the two branches of the optimum switching curve. Typical trajectories along with the optimum switching curve are shown in sketch (v). As is seen in sketch (v),



if  $|y/KB| < 1$ , there are trajectories which lead to the origin. If  $|y/KB| > 1$ , there are no trajectories which lead to the origin. Thus,  $|y/KB|$  must be kept less than unity if one is to maintain control over the unstable plant.

It is apparent that the optimum system for step inputs or initial conditions will use an ideal relay whose switching equation could be of the form

$$f(\epsilon) + \dot{\epsilon} = 0 \quad (67)$$

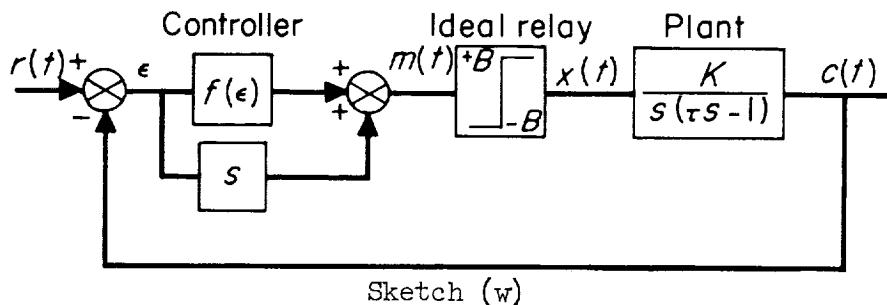
or

$$\epsilon + f(\dot{\epsilon}) = 0 \quad (68)$$

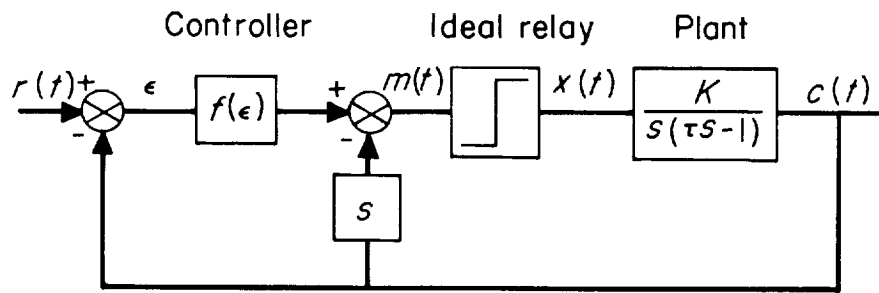
Equations (67) and (68) are the only possibilities if only functions of a single variable are considered.

The function  $f(\epsilon)$  in normalized form is the optimum curve shown in sketch (v) if one views  $x/KB\tau$  as  $\epsilon$ , the input to the function generator. Similarly,  $f(\dot{\epsilon})$  is the optimum switching curve shown in normalized form in sketch (v) if one views  $y/KB$  as  $\dot{\epsilon}$ , the input to the function generator. From practical standpoints it is believed that equation (67) is easier to fabricate since the function has the characteristic of a saturating device.

The optimum system for steps and initial conditions will therefore be of the form shown in sketch (w).



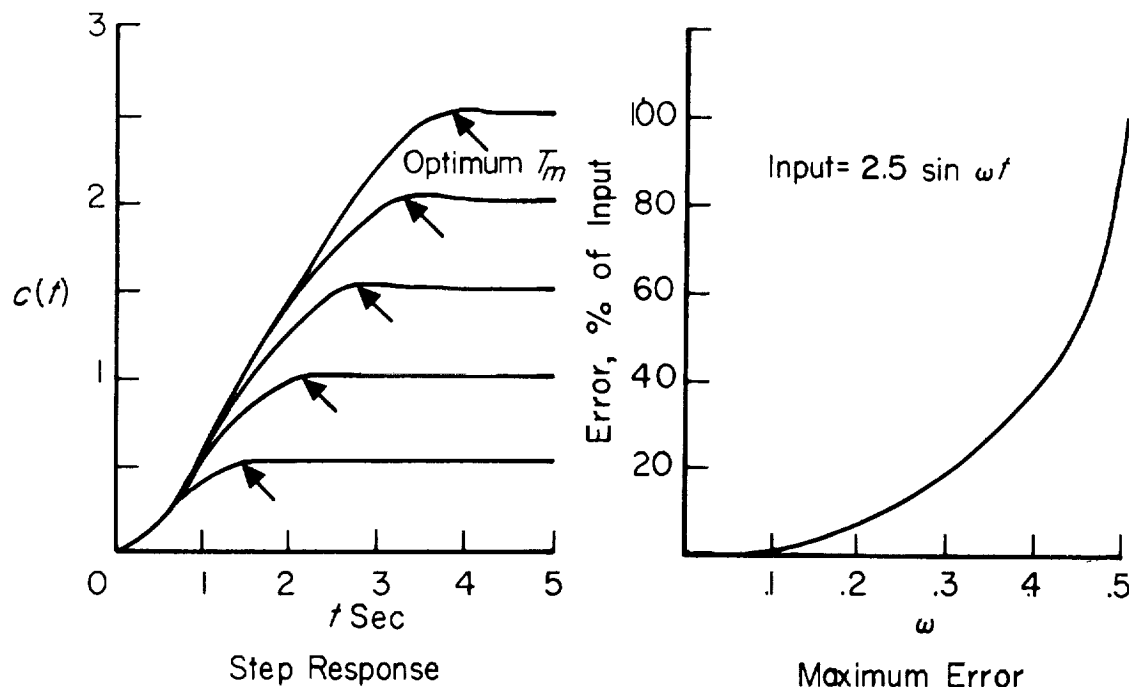
This system is probably near optimum for slowly changing inputs (i.e.,  $\dot{r}(t) \ll KB$ ); however, it certainly is nonoptimum for  $\dot{r}(t) > KB$  since this condition (if held for any length of time) could drive the system into instability. One way of preventing the input from being able to drive the system into instability is to use output rate feedback rather than error rate feedback. This alternative is shown in sketch (x).



Sketch (x)

The problem with this alternative is that zero error is not possible for changing inputs. The  $f(\epsilon)$  function shown in sketch (w) is very high gain for small errors, however, so the error for slowly changing inputs will be small. It shall be assumed that the system shown in sketch (x) is a near optimum system. The optimum system probably requires sensing of input derivatives and will not be determined here.

Some simulation tests were made of the system shown in sketch (x) for  $B = K = \tau = 1$ . The step response for several input magnitudes and the maximum error versus frequency for a constant amplitude sine wave input are shown in sketch (y). The step responses show the characteristic



Sketch (y)

of velocity limiting for the large inputs. This velocity limiting is a result of the nonlinear function of sketch (v). It must exist in order to prevent the system from becoming unstable. The maximum error versus

frequency curve shows the characteristic one would expect from sketch (v). That is, for  $\dot{r}(\max) = 2.5\omega < 1/4$ , the error is quite small. As the maximum rate of the input approaches unity and above, however, the maximum error increases rapidly.

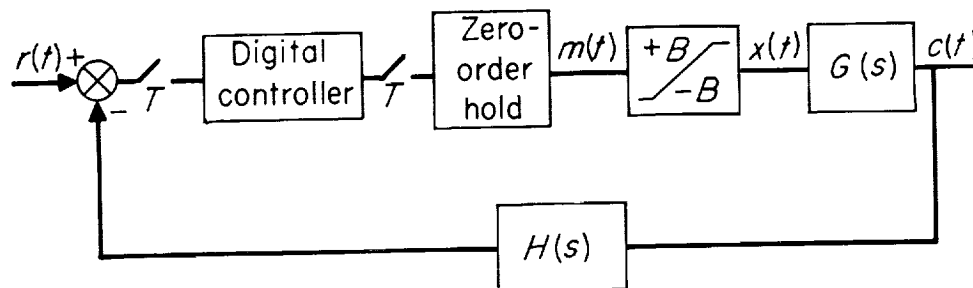
### Optimum Sampled-Data Systems

As has been shown, the optimum, continuous, saturated control system is directly obtained from the relay, or bang-bang, solution (at least, for stable plants). There still remain questions as to whether these systems are optimum for all inputs and certainly practical questions as to how one differentiates the error a number of times. The problem, at least, is very well understood in comparison to saturated sampled-data control systems. Perhaps because of the newness of sampled-data systems and the additional complexity caused by the introduction of sampling, very little is known about optimum, saturated, sampled-data systems.

It is the purpose of this section to consider the questions:  
 (1) What is a definition of the optimum, saturated, sampled-data system and what are its qualitative characteristics? (2) What is the optimum sampled-data system for a type 1 first-order plant? and (3) What are some of the pitfalls of trying to derive optimum systems for higher order plants? These three questions are considered in sequence.

An optimum, saturated, sampled-data system shall be defined as one which, regardless of the input or initial conditions, will maintain the error as small as possible with the operating sampling period,  $T$ , and with a limited input to the plant. The addition of the  $T$  is what makes the problem difficult. As  $T \rightarrow 0$  the sampled-data system approaches the continuous system. As  $T$  becomes very large, the sampled-data system performance with respect to varying inputs and response time to step inputs becomes very poor.

Consider the system shown in sketch (z).



Sketch (z)

Although there are many other sampled-data configurations possible, we shall restrict ourselves to this system where there is one sampler and a digital controller in the loop. The following information regarding the systems can be obtained from linear considerations. To have an optimum, one needs a transfer function of unity. Therefore

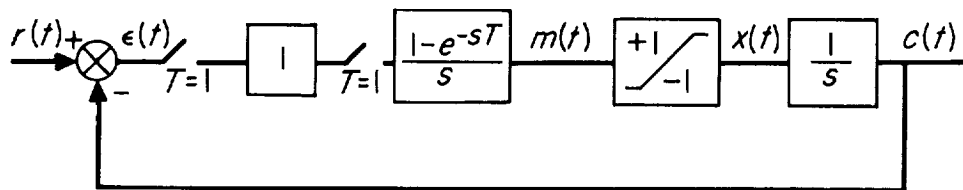
$$\left. \begin{aligned} H(s) &= 1 \\ T &\rightarrow 0 \\ \text{Gain of digital controller} &\rightarrow \infty \end{aligned} \right\} \quad (69)$$

If  $T$  is any finite value, the gain can never be infinite, so perhaps the optimum sampled-data system becomes a "finite settling time design" (refs. 7, 8, 9) with  $H(s) = 1$ . We must for relatively obvious reasons introduce the constraint of "zero ripple" (refs. 7, 9). The finite settling time design provides:

(1) The minimum number of sampling instants to restore the error to zero from an input transient. (Steps, ramps, etc., are permitted, depending upon the number of poles of  $G(s)$  at  $s = 0$ .) See, for example, reference 7 or 9.

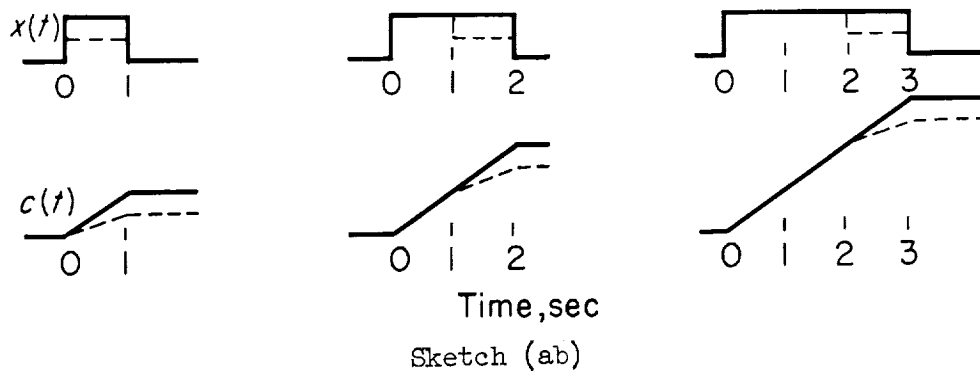
(2) The highest gain permissible at the prescribed sampling rate under the constraints of (1). Thus, for slowly varying inputs, the error will be very small. (Band width of the input should be small compared to the sampling frequency.)

Let us now examine a type 1, first-order plant with a finite settling time design. The block diagram is shown in sketch (aa) with values for  $T = 1.0$ ,  $B = 1$ ,  $K = 1$ .



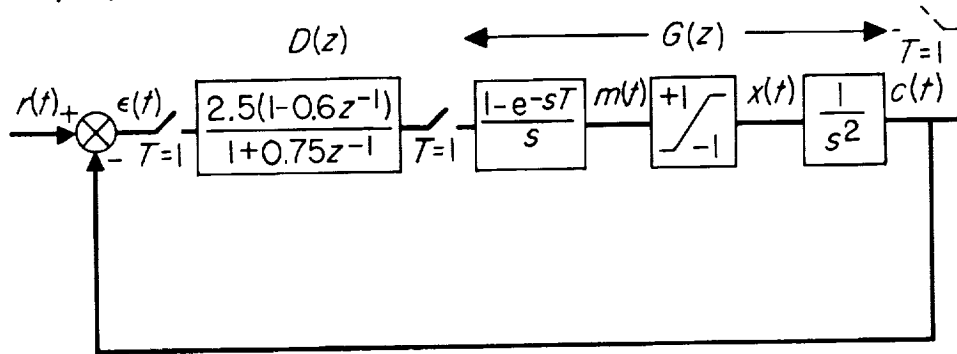
Sketch (aa)

Now consider a step input in the ranges shown in sketch (ab).



Quite obviously, from a study of the step responses, the finite settling time design is optimum with only a gain in the digital controller. A little further thought will show that this system should be optimum for arbitrary inputs, since the output will always move in a direction to reduce the error to zero (with no overshoot) in a minimum time based on the error information available at the present sampling instant. If  $T \rightarrow 0$  in this design, the gain of the digital controller approaches infinity and the over-all system approaches the continuous system optimum given or referred to in table VI.

Let us examine the design for the simple second-order plant where  $G(s) = 1/s^2$ ,  $T = 1$ ,  $B = 1$ . This design is given in sketch (ac).



The pulse transfer function

$$\frac{M(z)}{R(z)} = \frac{D(z)}{1 + G(z)D(z)} = 2.5(1 - 0.6z^{-1})(1 - z^{-1})^2 \quad (70)$$

and for a step of  $r(t)$  of magnitude  $R_0$

$$M(z) = R_0(2.5)(1 - 1.6z^{-1} + 0.6z^{-2}) \quad (71)$$

If  $R_0 > 1/(2.5)(1.6)$ , the value of  $m(t)$  exceeds unity (the limit level) and the pulse transfer function given in equation (70) is no longer valid. What happens for a large step of  $r(t)$ ? Certainly the input to the plant is saturated during the first sampling instant (see eq. (71)); therefore, one can compute what  $m(t)$  will be at the second sampling instant. This is most conveniently found by subtracting the value of the feedback quantity from the step response of  $D(z)$ .

$$m(1.0) = \left[ \frac{R_0 D(z)}{(1 - z^{-1})} - 2.5 \sum_{n=1}^{\infty} \frac{(n)^2 z^{-n}}{2} \right] \bigg|_{n=1} \quad (72)$$

where  $m(1.0)$  means the value of  $m$  between the first and second sampling instant.

The first term in the result must be expanded in a power series expression and the value of the coefficient of  $z^{-1}$  obtained. The second term is simply the step response of the plant and is valid so long as  $m > 1$ .

The result in carrying through the calculation is

$$m(1.0) = -0.875 R_0 - 1.25 \quad (73)$$

In other words, for large inputs, the input to the plant will always reverse at the first sampling instant. This is a result of having a pole of  $D(z)$  which is lightly damped and is the reason such systems were considered undesirable in chapter II of reference 1.

If the plant input reverses at the first sampling instant, it should be quite obvious from the arguments presented in chapter II of reference 1 that one cannot obtain anywhere close to optimum response with this linear digital controller. Thus, one recognizes that a nonlinear controller of some type must be used. Exactly what means one uses to design this controller is not recognized at present.

In summary, one can state the following about optimum saturated sampled-data systems:

1. They should be finite settling time designs in the linear region.
2. They will never be as good as the continuous system but will approach it as  $T \rightarrow 0$ .
3. Although the first-order case is very simple, extension even to second order probably requires a relatively complex nonlinear digital controller.

4. The determination of the nonlinear digital controller's mathematical operations does not appear to be a simple problem.

The fact that finite settling time designs are very sensitive to parameter variations (see refs. 8 and 9) and to the other difficulties listed probably means that optimum, saturated, sampled-data systems can never exist. Thus probably other designs are required for both nonlinear and linear operation; two such examples were given in chapter IV of reference 1.

#### APPROXIMATE OPTIMUM SYSTEMS

This section is devoted to the problem of designing an approximate optimum attitude autopilot for an aircraft. By approximate optimum is meant that for small or slowly varying inputs, if the aircraft has the capability of following a desired input (pitch attitude), the error will be essentially zero. For step inputs that may come about as a result of engaging the system, the response will be near optimum, that is, the error will be reduced to zero in a near minimum time. Disturbance inputs which may come about in actual practice as a result of gusts, will be neglected in this study; however, the resulting design which is a very high gain closed-loop system should have a good response to disturbances.

It will be assumed that the limiting in the system is a result of hydraulic valve bottoming in the control-surface actuator. For simplicity purposes it is further assumed that mechanical limits on the hydraulic valve are equivalent to control-surface rate limits, that is, the actuator is a perfect integrator. The control-surface rate limit will be assumed to be  $10^0$  per second. Control-surface position limiting will be neglected in this study.

Two transfer functions relating control-surface rate to pitch attitude are considered:

$$\frac{\theta}{\delta_e} = \frac{1.5625(s + 1)}{s^2 \left[ \left( \frac{s}{4} \right)^2 + 2 \left( \frac{0.1875}{4} \right) s + 1 \right]} \quad (74)$$

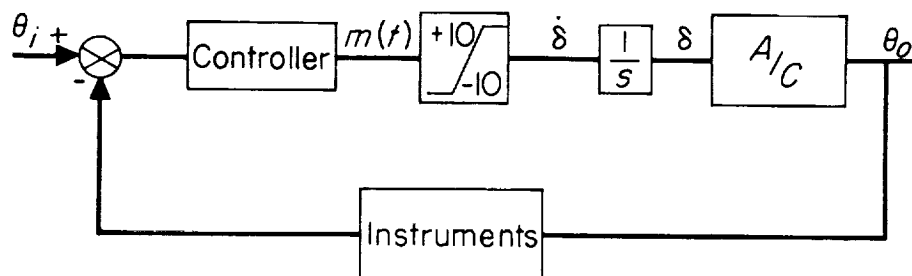
$$\frac{\theta}{\delta_e} = \frac{3.125(s + 1)}{s^2(0.183 s + 1)(0.683 s - 1)} \quad (75)$$

Equation (74) is the approximate linear relationship for the F-86 in normal configuration flying at  $M = 0.8$ ,  $h_p = 30,000$  feet. Equation (75)



is the approximate linear relationship for the F-86 with the c.g. shifted aft to give a statically unstable airframe. The flight condition is the same for both examples.

With the assumptions made, the problem reduces to one of designing the controller and instrumentation shown in the block diagram of sketch (ad).



Sketch (ad)

From considerations given in the previous section it is known that the desired instrument transfer function is unity and for very small errors, the linearized controller transfer function should be of the form

$$m(t) = K(1 + as + bs^2)\epsilon \quad (76)$$

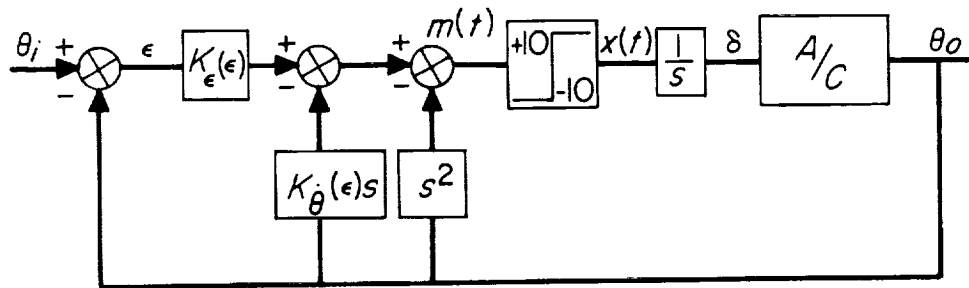
The open loop transfer function system given by sketch (ad) contains three zeros and four poles. Two of the zeros are a result of the controller; the other is from the aircraft transfer function. So long as the zeros are in the left half plane, in theory infinite gain is possible and the closed-loop transfer function  $\theta_o/\theta_i$  will be unity.

The problems associated with differentiation make it impossible to obtain the desired controller operations of equation (76). In general, one must add at least one pole for each differentiation operation. Furthermore, instruments must have dynamics associated with their measurements. The net result is that one cannot in practice obtain an open-loop transfer function with three zeros and only four poles. As a result, the high gain saturating control system will in practice have a limit cycle or chatter. It has been shown (ref. 10) that if this limit cycle is very high in frequency compared with the plant dynamics, one can closely approximate the infinite gain system.

For the unstable dynamics given by equation (75) it was shown in a previous example (for a simpler case) that if one uses error rate, the input is able to drive the system into instability. As a result of this system being higher order, we must neither use error rate nor error acceleration as stabilizing quantities. For our purposes we will use

output rate and output acceleration ( $\dot{\theta}_o, \ddot{\theta}_o$ ) for both aircraft transfer functions (eqs. (74), (75)). The result will be near optimum step response and a finite error to all inputs which are nonconstant.

These assumptions reduce the problem to the design of the system of sketch (ae) for the two plants of equations (74) and (75).



Sketch (ae)

The characteristic equations for the two systems are (under the assumption that the relay may be replaced by an equivalent gain, K):

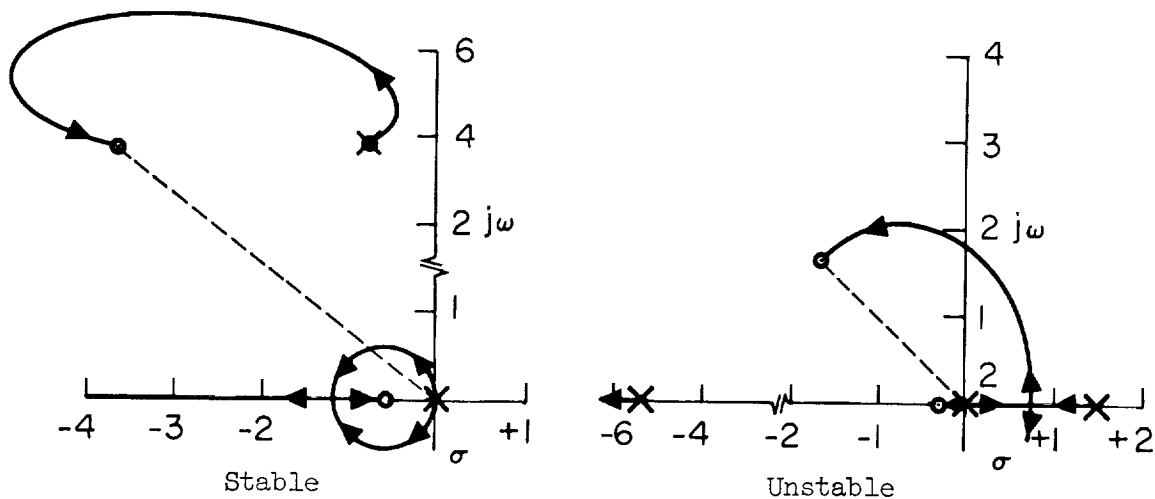
Stable airframe

$$1 + K[s^2 + K_{\dot{\theta}}(\epsilon)s + K_{\epsilon}(\epsilon)] \overbrace{\left\{ \frac{1.5625(s+1)}{s^2 \left[ \left(\frac{s}{4}\right)^2 + 2 \left(\frac{0.1875}{4}\right)s + 1 \right]} \right\}}^{G(s)} = 0 \quad (77)$$

Unstable airframe

$$1 + K[s^2 + K_{\dot{\theta}}(\epsilon)s + K_{\epsilon}(\epsilon)] \overbrace{\left[ \frac{3.125(s+1)}{s^2(0.183s+1)(0.683s-1)} \right]}^{G(s)} = 0 \quad (78)$$

The root locus as a function of equivalent limiter gain could be as shown in sketch (af) (for one value of  $K_{\dot{\theta}}(\epsilon)$  and  $K_{\epsilon}(\epsilon)$ ).

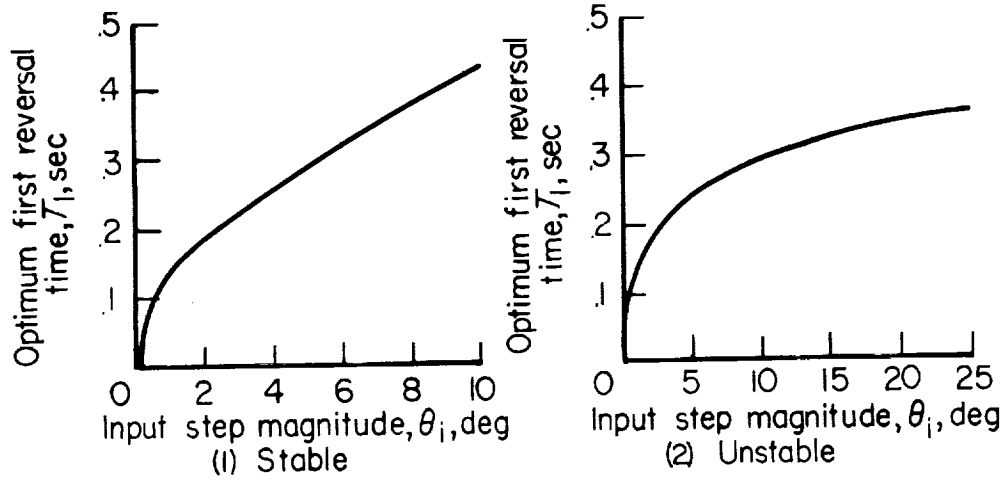


Sketch (af)

As can be seen from sketch (af), the stable system will be lightly damped and the unstable system will diverge for low values of equivalent limiter gain (or large step inputs). To have a good response for large inputs and a very fast system for small inputs the following two conditions will be imposed on the gains  $K_e(\epsilon)$  and  $K_\theta(\epsilon)$ .

1. They shall be such that the first reversal time for step inputs is the optimum.
2. They shall only be allowed to vary in a manner such that the damping ratio of the zeros determined by these quantities is a constant value of 0.7.

These two conditions are identical to those used for the third example of reference 1. To impose condition (1) it is necessary to have the optimum first reversal time as a function of the input step magnitude for the two example  $G(s)$  transfer functions. This information is available in normalized form in table IV for the stable airframe and in table V for the unstable airframe. Since tabulated data are not directly obtainable for the particular dynamics given by equations (74) and (75), one must interpolate. Nonlinear interpolation was used for these examples to increase the accuracy. The resultant optimum first reversal time curves are plotted in sketch (ag).



Sketch (ag)

To insure that the first reversal time of the system be optimum, it is necessary that  $m(t)$ , the input to the ideal relay, reverse sign at the optimum first reversal time given in sketch (ag) above. The equation for  $m(t)$ , the input to the ideal relay, derived from sketch (ae) is

$$m(t) = K_\epsilon(\epsilon)\epsilon + K_{\dot{\theta}}(\epsilon)\dot{\theta}_o + \ddot{\theta}_o \quad (79)$$

For step inputs and for a time of  $T_1$ , the first reversal time,

$$m(T_1) = 0 = K_\epsilon(\epsilon)[\theta_i(T_1) - \theta_o(T_1)] + K_{\dot{\theta}}(\epsilon)\dot{\theta}_o(T_1) + \ddot{\theta}_o(T_1) \quad (80)$$

The quantity  $\theta_i(T_1)$  is determined from sketch (ae) for the particular plant being studied. The quantities  $\theta_o(T_1)$ ,  $\dot{\theta}_o(T_1)$ ,  $\ddot{\theta}_o(T_1)$  are the responses of the output, output velocity, and output acceleration, respectively, for a step input  $x(t) = Bu(t)$  evaluated at  $t = T_1$ . Thus, to satisfy the first condition one simply finds an equation of form

$$K_\epsilon(\epsilon)[\epsilon(T_1)] + K_{\dot{\theta}}(\epsilon)\dot{\theta}_o(T_1) + \ddot{\theta}_o(T_1) = 0 \quad (81)$$

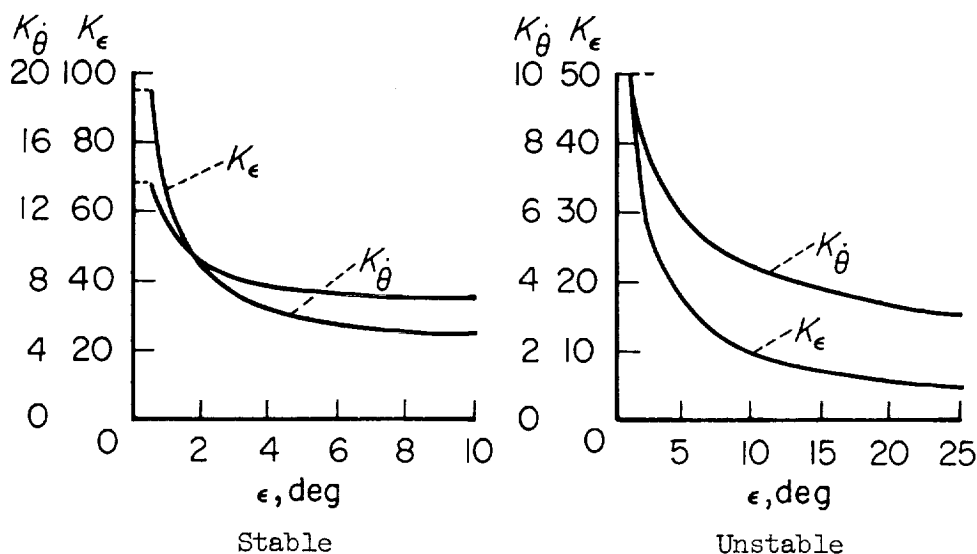
where  $\epsilon(T_1)$ ,  $\dot{\theta}_o(T_1)$ , and  $\ddot{\theta}_o(T_1)$  are constant for any particular value of  $T_1$ .

To satisfy the second condition one evaluates the damping ratios of the zeros determined by the quantities  $K_\epsilon(\epsilon)$  and  $K_{\dot{\theta}}(\epsilon)$ ; namely,

$$\zeta_o = \frac{1}{2} \frac{K_{\dot{\theta}}(\epsilon)}{\sqrt{K_\epsilon(\epsilon)}} \quad (82)$$

Setting  $\zeta_0 = 0.7$  in equation (82) one can solve for  $K_\theta(\epsilon)$  and substitute into equation (81) to obtain a solution for  $K_\epsilon(\epsilon)$  for each  $T_1$ . Alternatively one can plot  $K_\theta(\epsilon)$  versus  $K_\epsilon(\epsilon)$  from equation (82). Equation (81) is a straight line (for each  $T_1$ ) on this same plot. The intersection of the two curves is the desired value of  $K_\epsilon(\epsilon)$  and  $K_\theta(\epsilon)$  for the particular choice of  $T_1$ .

The functions of error were computed using this second alternative. The results are shown in sketch (ah).



Sketch (ah)

As can be seen, both  $K_\epsilon(\epsilon)$  and  $K_\theta(\epsilon)$  approach infinity as  $\epsilon \rightarrow 0$ . This means the zeros (and two of the closed-loop poles for the infinite gain system) go to infinity along the dotted line of sketch (af). For our purposes the gains were arbitrarily made constant for errors less than  $1/2^\circ$  for the stable case and less than  $1^\circ$  for the unstable case, as indicated by the dotted lines of sketch (ah). Thus, in the chatter mode of operation (see ref. 10), the system's responses with the stable airframe will be closely approximated by a second-order system with a natural

frequency of  $\sqrt{K_\epsilon(\epsilon)} = \sqrt{95} = 9.75$  radians per second, and a damping ratio of 0.7.

For the unstable airframe the system's response (in the chatter mode of operation) will be closely approximated by a second-order system of natural frequency  $\sqrt{K_\epsilon(\epsilon)} = 6.86$  radians per second and damping ratio of 0.7. In the chatter mode of operation both these systems are very high gain representing very "tight" attitude autopilots. The system will therefore closely follow any normal inputs and should have good response to gusts. In addition, the response of the system to step inputs over the range of inputs considered ( $10^\circ$  max in these cases) should be near optimum.

The two systems were simulated on an analog computer and the results of the simulation are presented in figures 7 and 8. Figure 7(a) shows the step responses of the system with stable aerodynamics. Figure 7(b) shows the control-surface deflection belonging to the step responses.

As can be seen from the control-surface deflection, the end of the response has a long tailing off exponential. The motion is a result of the fact that the system is an essentially infinite gain and one closed-loop pole of the system has a time constant of 1 second. As a result of the 1 second time constant zero in the aircraft transfer function, however, this motion does not occur on the output. Aerodynamically speaking, one could say that the pitch attitude is held constant while angle of attack and flight-path angle are slowly changing to take on new trim conditions.

The step responses are near optimum as can be seen by the arrows on the response representing the minimum response time for the particular input.

Figure 8(a) shows the step responses of the attitude control system for the unstable airframe. In this case also the response time is near the minimum. Figure 8(b) shows the control-surface motion for the various steps of figure 8(a). A comparison of figure 7(b) with 8(b) shows a significant difference in the character of the response. The difference is the first reversal time. As was mentioned earlier and shown in reference 1, the first reversal time for stable systems continues to increase with the input step magnitude. As the input goes to infinity,  $T_m - T_1$  approaches a constant. For unstable systems, however,  $T_1$  approaches a constant as  $T_m \rightarrow \infty$ . Since we have a near optimum system for both these examples, this fact is quite readily recognized from figures 7(b) and 8(b).

Figure 9 shows the maximum error as a function of frequency for the two systems. The input for both cases was  $10 \sin \omega t$ . As was hypothesized earlier, there is practically no error for low-frequency inputs. The error does increase with frequency, however, as a result of the use of output derivative feedbacks instead of error derivatives (for the lower frequencies) and a combination of this and limiting at the higher frequencies.

## CONCLUSIONS

It has been shown that the switch-time method for synthesis of nonlinear controllers to compensate for the effects of saturation on the input to the plant has very general application. It appears that only two restrictions must be imposed on the nonlinear controller design in order that one obtain a system which has a near optimum step response and well-damped response to other inputs. These restrictions are:

- (1) The first reversal time of the bounded variable shall be the optimum first reversal time of the optimum bang-bang control system.
- (2) The zeros (in the open loop transfer function) introduced by the controller should change (as a function of error) in such a manner that as the limiter equivalent gain is decreased, the poles of the closed-loop system remain in well-damped regions of the  $s$  plane.

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To meet restriction (1) it is necessary to have data on the optimum first reversal time as a function of input step magnitude. This report has presented such data for several additional plant transfer functions over those considered in reference 1. In addition, an approximate method for determining such data has been presented. The approximate method has been shown to be reasonably accurate for plants of at least fourth order.

To meet restriction (2) it has been found that if only one zero is introduced by the controller, then as the error increases, this zero should move toward the origin on the negative real axis. For controllers with two zeros, it has been found that if the zeros move toward the origin along a line of constant damping ratio (as the error increases) good characteristics are obtained. Systems with three or more zeros introduced by the controller have not been studied.

It has been shown that if the plant has a zero in the left half plane, the system's response can be made faster than if the zero is neglected. This solution requires some of the plant variables to change as the error and error derivatives remain constant; however, examples have shown that it is a practical solution.

The effects of an unstable pole are to force the controller to keep certain output derivatives from exceeding certain limits; otherwise, control over the plant is lost and the system response diverges. It is interesting to note that if the plant has stable poles, some of the output derivatives are bounded because the plant input is bounded. Thus there are not significant differences between stable and unstable plants in this respect.

With regard to optimum continuous systems it has been shown that with certain restrictions they are derivable from the optimum relay solution. Optimum sampled-data systems, however, need considerably more research before one can derive the optimum controller except for extremely simple plants.

One can generally conclude that saturation in single variable control systems is one type of nonlinearity that lends itself to analytical treatment such that its effects in any particular system can be well understood. Furthermore, it can be concluded that if the effects are undesirable, such

as to make the system unstable or oscillatory for large inputs, the switch time method can be used to compute compensating nonlinear functions in the controller which remove these undesirable characteristics.

Ames Research Center  
National Aeronautics and Space Administration  
Moffett Field, Calif., April 8, 1960

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TABLE I.- TRANSFER FUNCTIONS OF PLANTS CONSIDERED

Case	$G(s)$	Type
1	$\frac{K}{s}$	1 - first order
2	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$	1 - second order
3	$\frac{K}{s[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	1 - third order
4	$\frac{K}{s^2}$	2 - second order
5	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$	2 - third order
6	$\frac{K}{s^2[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	2 - fourth order
7	$\frac{K}{s^3}$	3 - third order
8	$\frac{K}{s(\tau s - 1)}$	1 - second order, unstable
9	$\frac{K(\tau_1 s + 1)}{s[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	1 - third order (with zero)
10	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$	1 - third order (with zero) unstable
11	$\frac{K(\tau_1 s + 1)}{s^2[(s^2/\omega_n^2) + (2\zeta s/\omega_n) + 1]}$	2 - fourth order (with zero)
12	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)}$	2 - fourth order (with zero) unstable

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TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT WITH ZERO

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	b/B	$R_0 \omega_n / BK$
$\xi = 0.10; \omega_{n1} = 0$					$\xi = 0.10; \omega_{n1} = 1.0$					$\xi = 0.25; \omega_{n1} = 0$				
0.5	1.2852	1.7423		0.1719	8.0	8.6151	9.1176		8.8875	9.0	9.7561	10.8091		9.2970
1.0	2.0402	2.8951		.8148	9.0	9.4434	10.3296		10.4429	10.0	10.7226	11.7792		10.3540
1.5	2.5065	3.7169		1.7038	10.0	10.2668	11.1636		11.5901	11.0	11.6898	12.7270		11.3473
2.0	2.8570	4.3717		2.6577	11.0	11.2339	11.7699		12.3021	12.0	12.6954	13.6971		12.3062
2.5	3.1743	4.9161		3.5674	12.0	12.2628	12.4510		12.8853	13.0	13.7153	14.7038		13.2732
3.0	3.5069	5.3616		4.3479	13.0	13.4320	13.5429		13.6790	14.0	14.7294	15.7280		14.2691
3.5	3.8828	5.7119		4.9463	14.0	14.5226	14.8861		14.6430	15.0	15.7295	16.7447		15.2858
4.0	4.3080	5.9868		5.3708	15.0	15.4667	16.0901		16.1567	16.0	16.7212	17.7450		16.3026
5.0	5.2790	6.4610		5.9029	16.0	16.3667	17.0602		17.3228	17.0	17.7139	18.7353		17.3074
6.0	6.4358	7.1970		6.3254	17.0	17.3146	17.8648		18.2355	18.0	18.7124	19.7264		18.3015
7.0	7.7642	8.6216		7.0932	18.0	18.3275	18.6692		19.0142	19.0	19.7155	20.7244		19.2933
8.0	8.8011	10.0313		8.4292	19.0	19.3969	19.6503		19.8565	20.0	20.7192	21.7281		20.2897
9.0	9.6275	11.1206		9.8656	20.0	20.4594	20.8075		20.8874	$\xi = 0.25; \omega_{n1} = 0.5$				
10.0	10.4707	11.9576		11.0161	21.0	21.4524	21.9513		22.0464	.1	.2129	0.3721	-1	.5741
11.0	11.4190	12.6723		11.8343	22.0	22.4017	22.9765		23.1730	.2	.4413	.1339		.2564
12.0	12.4878	13.4908		12.5151	23.0	23.3603	23.8862		24.1686	.3	.6707	.2665		.6253
13.0	13.6423	14.6078		13.3232	24.0	24.3599	24.7737		25.0619	.4	.8913	.4173		.1174
14.0	14.7141	15.8501		14.4219	25.0	25.3870	25.7315		25.9574	.5	1.0983	.5755		.1894
15.0	15.6500	16.9637		15.6636	26.0	26.4246	26.7910		26.9419	.6	1.2905	.7353		.2772
16.0	16.5515	17.9120		16.8091	27.0	27.4330	27.8792		28.0131	.7	1.4682	.8933		.3785
17.0	17.4994	18.7624		17.7637	28.0	28.4110	28.9166		29.0946	.8	1.6324	1.0560		.4912
18.0	18.5203	19.6399		18.5994	29.0	29.3844	29.8849		30.1162	.9	1.7812	1.2161		.6136
19.0	19.5952	20.5550		19.4646	30.0	30.3749	30.8213		31.0725	1.0	1.9166	1.3761		.7441
20.0	20.6516	21.7745		20.4713	$\xi = 0.10; \omega_{n1} = 2.0$					1.5	2.4625	2.8971		1.4721
$\xi = 0.10; \omega_{n1} = 0.5$					.1	.2019	0.1120	-1	.2048	2.0	2.8950	3.5335		2.2434
.1	.4537	.1621	-1	.5925	.2	.4034	.4311	-1	.3283	2.5	3.2899	4.0648		2.9870
.2	.6958	.3289		.6860	.3	.5987	.9247	-1	.1862	3.0	3.6811	4.5250		3.6628
.3	.9298	.5233		.1319	.4	.7840	1.557		.3273	3.5	4.0908	4.9345		4.2528
.4	1.1489	.7321		.2171	.5	.9573	.2294		.5014	4.0	4.5266	5.3121		4.7589
.5	1.3511	.9470		.3224	.6	1.1182	.3106		.7030	4.5	4.9908	5.6792		5.1976
.6	1.5335	1.6125		.4455	.7	1.2672	.3970		.9269	5.0	5.4834	6.0600		5.5931
.7	1.6929	1.8698		.5840	.8	1.4054	.4868		1.1682	5.5	5.9834	6.4548		6.3638
.8	1.8337	2.1033		.7359	.9	1.5340	.5783		1.4226	6.0	6.4855	6.9548		7.2576
.9	1.9594	2.3178		.8990	1.0	1.6543	.6705		1.6867	6.5	6.9921	7.2576		8.3097
1.0	2.4397	3.2012		1.8217	1.5	2.1611	2.3814		3.0592	7.0	7.4993	8.0562		10.4721
1.5	2.7921	3.8955		2.8112	2.0	2.4953	3.4878		4.4973	8.0	8.6921	9.1939		12.4170
2.0	3.1063	4.4687		3.7560	$\xi = 0.10; \omega_{n1} = 4.0$					10.0	10.6233	11.2187		14.3797
2.5	3.4356	4.9298		4.5585	.1	.1995	.9868	-1	.3997	12.0	12.6040	13.1249		16.4159
3.0	3.8124	5.2764		5.1516	.2	.3944	.3808	-1	.7580	14.0	14.6378	15.1553		18.4137
3.5	4.2413	5.5312		5.5487	.3	.5801	.8198	-1	.3478	16.0	16.6305	17.1769		20.4013
4.0	5.2116	5.9412		6.0179	.4	.7543	.1386		.6001	18.0	18.6214	19.1565		22.4074
4.5	6.3475	6.5963		6.4012	.5	.9160	.2051		.9044	20.0	20.6280	21.1573		24.4093
5.0	7.6836	8.0448		7.1776	.6	1.0654	.2789		1.2503	22.0	22.6285	23.1645		26.4062
5.5	8.7346	9.5183		8.5491	.7	1.2036	.3580		1.6283	24.0	24.6299	25.1611		28.4069
6.0	9.5627	10.6417		10.0163	.8	1.3318	.4405		2.0303	26.0	26.6268	27.1599		30.4077
6.5	10.4060	11.4779		11.1660	.9	1.4513	.5251		2.4492	28.0	28.6274	29.1617		
7.0	11.3529	12.1625		11.9567	1.0	1.5635	.6107		2.8792	30.0	30.6268	31.1614		
7.5	12.4150	12.9427		12.6128	1.5	2.0460	2.1422		5.0502	$\xi = 0.25; \omega_{n1} = 1.0$				
8.0	13.5678	14.0530		13.4173	$\xi = 0.10; \omega_{n1} = 8.0$					.1	.2037	.1415	-1	.1043
8.5	14.6456	15.3235		14.5323	.1	.1983	.9657	-1	.7896	.2	.4097	.5262	-1	.4290
9.0	15.5847	16.4620		15.7926	.2	.3898	.3729	-1	.3085	.3	.6113	.1091		.9781
9.5	16.4865	17.4164		16.9434	.3	.5708	.8035	-1	.6720	.4	.8039	.1760		1.740
10.0	17.4334	18.2539		17.8872	.4	.7392	.1360		1.1487	.5	.9853	.2548		.2695
10.5	18.4514	19.1109		18.7081	.5	.8949	.2015		1.7168	.6	1.1547	.3362		.3815
11.0	19.5245	20.1165		19.5676	.6	1.0383	.2743		2.3559	.7	1.3127	.4201		.5074
11.5	20.5828	21.2459		20.5804	.7	1.1708	.3524		3.0483	.8	1.4601	.5049		.6448
12.0	21.7202	22.3598		21.7088	.8	1.2938	.4341		3.7768	.9	1.5981	.5893		.7912
12.5	22.9202	23.3598		22.8194	.9	1.4085	.5179		4.5349	1.0	1.7278	.6727		.9449
13.0	24.1794	24.2835		23.8247	1.0	1.5163	.6028		5.3060	1.5	2.2855	2.3408		1.7697
13.5	24.4774	25.1955		24.7407	1.5	1.9831	2.0946		9.1284	2.0	2.7299	3.0687		2.6090
14.0	25.5108	26.1725		25.6508	$\xi = 0.10; \omega_{n1} = 16.0$					2.5	3.1254	3.6515		3.4008
14.5	26.5476	27.2261		26.6309	.1	.1977	.9665	-1	.1569	3.0	3.5168	4.1355		4.1019
15.0	27.5538	28.2938		27.6863	.2	.3876	.3733	-1	.6097	3.5	3.9265	4.5449		4.6919
15.5	28.5305	29.3165		28.7555	.3	.5661	.8045	-1	1.3211	4.0	4.3630	4.9021		5.1762
16.0	29.5039	30.2861		29.7782	.4	.7317	.1362		2.2474	4.5	4.8268	5.2338		5.5803
16.5	30.4956	31.2358		30.7446	.5	.8843	.2018		3.3447	5.0	5.3160	5.5700		5.9379
$\xi = 0.10; \omega_{n1} = 1.0$					.6	1.0247	.2748		4.5728	5.5	5.8260	5.9048		6.2528
.1	.2067	.1746	-1	.1075	.7	1.1544	.3532		5.8971	6.0	6.3660	6.3848		6.6528
.2	.4211	.6658	-1	.4551	.8	1.2746	.4352		7.2885	6.5	6.9696	6.8050		7.0499
.3	.6344	.1409		.1065	.9	1.3870	.5194		8.7231	7.0	7.4696	7.4850		7.4599
.4	.8399	.2337		.1939	1.0	1.4925	.6046		10.1811	7.5	8.0205	8.0707		7.8697
.5	1.0335	.3392		.3058	$\xi = 0.25; \omega_{n1} = 0$					8.0	8.5205	8.6707		8.2833
.6	1.2135	.4529		.4394	.5	1.2567	1.6609		.1475	10.0	10.4566	10.7366		10.8233
.7	1.3801	.5715		.5914	1.0	2.0301	2.7264		.6662	12.0	12.4336	12.6106		12.7434
.8	1.5340	.6926		.7586	1.5	2.5573	3.4842		1.3695	14.0	14.4672	14.6394		14.7050
.9	1.6766	.8145		.9379	2.0	2.9834	4.0876		2.1208	16.0	16.4616	16.6740		16.7507
1.0	1.8093	.9358		1.1265	2.5	3.3754	4.6002		2.8495	18.0	18.4519	18.6485		18.7447
1.5	2.3196	2.7934		2.1542	3.0	3.7672	5.0517		3.5174	20.0	20.4582	20.6468		20.7304
2.0	2.6722	3.5907		3.2463	3.5	4.1769	5.4612		4.1074	22.0	22.4591	22.6570		22.7388
2.5	2.9728	4.2482		4.3026	4.0	4.6129	5.8462		4.6203	24.0	24.4564	24.6532		24.7404
3.0	3.2879	4.7633		5.1876	4.5	5.0733	6							

TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT  
 WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0 \omega_n / BK$
$\xi = 0.25; \omega_n T_1 = 2.0$					$\xi = 0.50; \omega_n T_1 = 0$					$\xi = 0.50; \omega_n T_1 = 4.0$				
0.8		1.3359	0.3691	1.0023	16.0	16.9505	17.7223		15.8214	0.1		0.1923	0.7352	-2
0.9		1.4594	0.4354	1.2115	17.0	17.9505	18.7224		16.3215	0.2		0.3693	0.6655	-1
1.0		1.5761	0.5018	1.4276	18.0	18.9504	19.7224		17.8216	0.3		0.5318	0.5450	-1
1.5		2.0868	0.8157	2.5446	19.0	19.9504	20.7223		18.8216	0.4		0.6815	0.6834	-1
2.0	2.5260	2.6653		3.6133	20.0	20.9503	21.7223		19.8216	0.5		0.8200	0.1263	
2.5	2.9136	3.3940		4.5667						0.6		0.9492	0.1670	
3.0	3.2958	3.9519		5.3602	$\xi = 0.50; \omega_n T_1 = 0.5$					0.7		1.0705	0.2094	1.1672
3.5	3.7056	4.3609		5.9496	0.1		0.2079	0.2668	-1	0.8		1.1854	0.2528	1.4258
4.0	4.1502	4.6548		6.3544	0.2		0.4231	0.9272	-1	0.9		1.2948	0.2965	1.6912
4.5	4.6216	4.8841		6.6410	0.3		0.6354	1.797	-1	1.0		1.3997	0.3401	1.9605
5.0		5.1115	0.9924	6.8732	0.4		0.8387	2.754	-1	1.5		1.6806	0.5451	3.2999
6.0		6.1705	0.7848	7.3992	0.5		1.0307	3.729	-1	2.0		2.3251	0.7122	4.5237
7.0		7.2745	0.7820	8.2896	0.6		1.2108	4.686	-1	2.5		2.7607	0.8310	5.5632
8.0		8.3166	0.8986	9.4607	0.7		1.3796	5.609	-1	3.0		3.2007	0.9006	6.4018
10.0	10.2526	10.2848		11.7797	0.8		1.5383	6.491	-1	3.5		3.6522	0.9271	7.0560
12.0		12.2291	0.9226	13.6161	0.9		1.6879	7.328	-1	4.0		4.1183	0.9207	7.5645
14.0		14.2627	0.9183	15.5738	1.0		1.8295	8.121	-1	4.5		4.5993	0.8941	7.9772
16.0		16.2968	0.9545	17.6521	1.5	2.4475	2.5179		1.1229	5.0		5.0934	0.8996	8.3450
18.0		18.2471	0.9402	19.6334	2.0	2.9663	3.1456		1.7130	5.5		5.5971	0.8268	8.7100
20.0		20.2535	0.9330	21.6126	2.5	3.4412	3.6896		2.3073	6.0		6.1060	0.8014	9.0995
22.0		22.2543	0.9414	23.6285	3.0	3.9003	4.1871		2.8866	7.0		7.1259	0.7788	9.8991
24.0		24.2516	0.9404	25.6292	4.0	4.8228	5.1205		3.9748	8.0		8.1365	0.7825	10.9933
26.0		26.2524	0.9377	27.6230	5.0	5.7820	6.0465		4.9824	9.0		9.1375	0.7935	12.0364
28.0		28.2531	0.9393	29.6225	6.0	6.7776	7.0093		5.9542	10.0		10.1344	0.8008	13.0687
30.0		30.2525	0.9386	31.6267	8.0	8.8020	9.0229		7.9188	12.0		12.1306	0.8011	15.0738
$\xi = 0.25; \omega_n T_1 = 4.0$					10.0	10.8057	11.0371		9.9256	14.0		14.1313	0.7986	17.0630
0.1		0.1967	0.8880	-2	12.0	12.8020	13.0336		11.9295	16.0		16.1317	0.7988	19.0635
0.2		0.3842	0.3341	-1	14.0	14.8020	15.0321		13.9282	18.0		18.1316	0.7991	21.0650
0.3		0.5600	0.7041	-1	16.0	16.8025	17.0327		15.9278	20.0		20.1316	0.7991	23.0647
0.4		0.7233	0.1169		18.0	18.8024	19.0329		17.9280	22.0		22.1316	0.7990	25.0646
0.5		0.8745	0.1705		20.0	20.8024	21.0327		19.9280	24.0		24.1316	0.7990	27.0646
0.6		1.0147	0.2290	1.1015	22.0	22.8024	23.0327		21.9280	$\xi = 0.50; \omega_n T_1 = 8.0$				
0.7		1.1450	0.2910	1.4189	24.0	24.8024	25.0328		23.9280	0.1		0.1912	0.8061	-2
0.8		1.2670	0.3551	1.7533	26.0	26.8024	27.0328		25.9280	0.2		0.3653	0.2924	-1
0.9		1.3817	0.4203	2.0994	$\xi = 0.50; \omega_n T_1 = 1.0$					0.3		0.5239	0.9882	-1
1.0		1.4903	0.4858	2.4588	0.1		0.1991	0.9011	-2	0.4		0.6690	0.9706	-1
1.5		1.9708	0.7986	4.2237	0.2		0.3931	0.3232	-1	0.5		0.8028	1.1389	1.3083
2.0	2.3914	2.6112		5.6283	0.3		0.5790	0.6513	-1	0.6		0.9273	1.1838	1.7433
2.5	2.7307	2.7877		7.3172	0.4		0.7552	1.039	-1	0.7		1.0440	1.2307	2.2016
$\xi = 0.25; \omega_n T_1 = 8.0$					0.5		0.9213	1.460	-1	0.8		1.1545	1.2787	2.6751
0.1		0.1955	0.9030	-2	0.6		1.0778	1.898	-1	0.9		1.2599	1.3271	3.1572
0.2		0.3799	0.3400	-1	0.7		1.2254	2.342	-1	1.0		1.3610	1.3755	3.6428
0.3		0.5512	0.7170	-1	0.8		1.3652	2.785	-1	1.5		1.8269	0.6036	6.0016
0.4		0.7093	0.1192		0.9		1.4979	3.220	-1	2.0		2.2620	0.7900	8.0579
0.5		0.8551	0.1739	1.5365	1.0		1.6245	3.645	-1	2.5		2.6920	0.9227	9.6988
0.6		0.9898	0.2339	2.0817	1.5		2.1906	5.548	-1	3.0	3.1291	3.1341		10.8759
0.7		1.1149	0.2975	2.6651	2.0		2.6896	7.015	-1	3.5	3.5775	3.8195		11.6645
0.8		1.2320	0.3634	3.2745	2.5		3.1580	8.024	-1	4.0	4.0447	4.2290		12.1397
0.9		1.3422	0.4305	3.9014	3.0		3.6155	9.606	-1	4.5		4.5279	0.9933	12.4188
1.0		1.4467	0.4979	4.5369	3.5		4.0738	11.172	-1	5.0		5.0235	0.9547	12.6144
1.5		1.9116	0.8210	7.6565	4.0		4.5378	12.724	-1	5.5		5.5285	0.9180	12.8153
2.0	2.3079	3.0752		10.4393	5.0		5.4980	14.264	-1	6.0		6.0387	0.8896	13.0783
$\xi = 0.25; \omega_n T_1 = 16.0$					6.0		6.4596	15.775	-1	6.5		6.5499	0.8720	13.4261
0.1		0.1949	0.9215	-2	8.0		8.5209	17.261	-1	7.0		7.0596	0.8643	13.8551
0.2		0.3777	0.3470	-1	10.0		10.5238	18.770	-1	8.0		8.0700	0.8685	14.8778
0.3		0.5468	0.7320	-1	12.0		12.5200	20.252	-1	9.0		9.0705	0.8808	15.9759
0.4		0.7023	0.1217		14.0		14.5201	21.751	-1	10.0		10.0671	0.8889	17.0445
0.5		0.8454	0.1777	2.9974	16.0		16.5206	23.254	-1	12.0		12.0632	0.8893	19.0513
0.6		0.9774	0.2390	4.0467	18.0		18.5205	24.755	-1	14.0		14.0641	0.8865	21.0276
0.7		1.0999	0.3040	5.1645	20.0		20.5205	26.255	-1	16.0		16.0645	0.8868	23.0297
0.8		1.2145	0.3714	6.3280	22.0		22.5205	27.755	-1	18.0		18.0643	0.8871	25.0326
0.9		1.3225	0.4401	7.5189	$\xi = 0.50; \omega_n T_1 = 2.0$					20.0		20.0643	0.8870	27.0320
1.0		1.4249	0.5092	8.7221	0.1		0.1946	0.781	-2	22.0		22.0643	0.8870	29.0317
1.5		1.8821	0.8402	14.5604	0.2		0.3773	1.2453	-1	$\xi = 0.50; \omega_n T_1 = 16.0$				
$\xi = 0.50; \omega_n T_1 = 0$					0.3		0.5477	1.7000	-1	0.1		0.1907	0.8921	-2
0.5	1.2188	1.5561		1.1186	0.4		0.7065	2.1511	-1	0.2		0.3633	0.3091	-1
1.0	2.0141	2.5362		5.080	0.5		0.8546	2.6078	-1	0.3		0.5199	0.6326	-1
1.5	2.6105	3.2559		1.0350	0.6		1.0133	3.0645	-1	0.4		0.6688	1.0265	1.7796
2.0	3.1183	3.8491		1.6126	0.7		1.1239	3.5211	-1	0.5		0.7943	1.469	2.5565
2.5	3.5879	4.3745		2.1988	0.8		1.2476	3.9777	-1	0.6		0.9164	1.945	3.3955
3.0	4.0446	4.8627		2.7734	0.9		1.3653	4.4344	-1	0.7		1.0310	2.442	4.2757
3.5	4.5019	5.3316		3.3279	1.0		1.4781	4.8911	-1	0.8		1.1393	2.950	5.1810
4.0	4.9666	5.7935		3.8604	1.5		1.9898	6.830	-1	0.9		1.2427	3.463	6.0989
5.0	5.9277	6.7274		4.8719	2.0		2.4542	8.765	-1	1.0		1.3420	3.976	7.0196
6.0	6.9255	7.6985		5.8476	2.5		2.9018	10.700	-1	1.5		1.8008	6.395	11.4320
7.0	7.9391	8.7015		6.8234	3.0		3.3482	12.635	-1	2.0		2.2314	8.834	15.1676
8.0	8.9507	9.7149		7.8135	3.5		3.7975	14.570	-1	2.5		2.6587	11.764	17.9924
9.0	9.9548	10.7241		8.8146	4.0		4.2675	16.505	-1	$\xi = 0.75; \omega_n T_1 = 0$				
10.0	10.9537	11.7264		9.8191	4.5		4.7372	18.440	-1	0.5	1.1893	1.4772		0.8666
11.0	11.9513	12.7249		10.8222	5.0		5.2070	20.375	-1	1.0	2.0013	2.4113		4.086
12.0	12.9499	13.7228		11.8230	5.5		5.6768	22.310	-1	1.5	2.6471	3.1267		0.3225
13.0	13.9496	14.7217		12.8225	6.0		6.1466	24.245	-1	2.0	3.2129	3.7364		1.3106
14.0	14.9500	15.7217		13.8218	6.5		6.6164	26.180	-1	2.5	3.7382	4.2898		1.8134
15.0	15.9503	16.7220		14.8214	7.0		7.0862	28.115	-1	3.0	4.2431	4.8121		2.3259
					7.5		7.5560	30.050	-1	3.5	4.7385	5.3176		2.8405



TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT  
WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	b/B	$R_0 \omega_n / BK$
$\zeta = 1.02; \omega_{n1} = 4.0$					$\zeta = 1.50; \omega_{n1} = 0$					$\zeta = 1.50; \omega_{n1} = 2.0$				
0.8	1.0774	0.1249		1.0220	7.5	9.5329	9.7958		5.7300	22.0	22.3884	-0.2238		21.1641
.9	1.1816	.1446		1.1968	8.0	10.0458	10.3089		6.2172	24.0	24.3885	-.2238		23.1640
1.0	1.2834	.1641		1.3730	8.5	10.5565	10.8194		6.7064	26.0	26.3885	-.2238		25.1640
1.5	1.7718	.2547		2.2469	9.0	11.0636	11.3266		7.1994					
2.0	2.2470	.3299		3.0727	9.5	11.5708	11.8338		7.6922					
3.0	3.2007	.4337		4.5340	10.0	12.0769	12.3399		8.1862					
4.0	4.1715	.4898		5.7878										
6.0	6.1484	.5312		7.9763										
8.0	8.1432	.5401		10.0174										
10.0	10.1421	.5420		12.0258										
12.0	12.1419	.5423		14.0274										
14.0	14.1418	.5424		16.0277										
16.0	16.1418	.5424		18.0278										
18.0	18.1418	.5424		20.0278										
$\zeta = 1.02; \omega_{n1} = 8.0$					$\zeta = 1.50; \omega_{n1} = 0.5$					$\zeta = 1.50; \omega_{n1} = 4.0$				
.1	.1834	.0635	-2	.6706	-1	.1916	-0.7579	-2	.4570	.1	.1762	.2404	-2	.3145
.2	.3413	.2145	-1	.2304	.2	.3711	-.2378	-1	.1697	.2	.3273	.7805	-2	.1039
.3	.4818	.4186	-1	.4532	.3	.5427	-.4298	-1	.3581	.3	.4598	.1474	-1	.1992
.4	.6099	.6549	-1	.7140	.4	.7066	-.6251	-1	.6019	.4	.5817	.2251	-1	.3083
.5	.7291	.9109	-1	.9996	.5	.8700	-.8106	-1	.8944	.5	.6966	.3073	-1	.4263
.6	.8416	.1178		1.3011	.6	1.0279	-.9806	-1	.1231	.6	.8066	.3917	-1	.5500
.7	.9491	.1452		1.6123	.7	1.1827	-.1133		.1607	.7	.9133	.4768	-1	.6774
.8	1.0528	.1727		1.9288	.8	1.3347	-.1368		.2019	.8	1.0175	.5618	-1	.8072
.9	1.1536	.2001		2.2474	.9	1.4842	-.1586		.2465	.9	1.1199	.6458	-1	.9384
1.0	1.2522	.2272		2.5657	1.0	1.6314	-.1790		.2942	1.0	1.2210	.7284	-1	1.0704
1.5	1.7273	.3535		4.1010	1.5	2.3365	-.1837		.5717	1.5	1.7167	.1113		1.7284
2.0	2.1931	.4588		5.4777	2.0	2.9988	-.2016		.9004	2.0	2.2072	.1442		2.3696
2.5	2.6614	.5417		6.6725	2.5	4.2305	-.2172		1.6609	2.5	3.1901	.1949		3.5896
3.0	3.1355	.6044		7.6997	3.0	5.3795	-.2233		2.5088	3.0	4.1782	.2300		4.7417
3.5	3.6156	.6504		8.5875	3.5	7.5418	-.2277		4.3443	4.0	6.1648	.2705		6.9173
4.0	4.1010	.6834		9.3662	4.0	9.6144	-.2292		6.2710	6.0	8.1587	.2895		8.9994
5.0	5.0830	.7228		10.0998	4.5	11.6475	-.2297		8.2376	8.0	10.1559	.2984		11.0378
6.0	6.0742	.7416		11.8587	5.0	13.6629	-.2300		10.2221	10.0	12.1545	.3026		13.0557
8.0	8.0682	.7542		13.9655	6.0	15.6700	-.2301		12.2150	12.0	14.1539	.3044		15.0638
10.0	10.0669	.7568		15.9872	8.0	17.6733	-.2302		14.2117	14.0	16.1536	.3054		17.0679
12.0	12.0667	.7573		17.9914	10.0	19.6748	-.2302		16.2101	16.0	18.1535	.3058		19.0697
14.0	14.0666	.7574		19.9923	12.0	21.6755	-.2302		18.2094	18.0	20.1535	.3060		21.0705
16.0	16.0666	.7574		21.9925	14.0	23.6759	-.2302		20.2090	20.0	22.1534	.3061		23.0709
18.0	18.0666	.7574		23.9925	16.0	25.6760	-.2302		22.2088	22.0	24.1534	.3061		25.0710
20.0	20.0666	.7574		25.9923	18.0	27.6761	-.2302		24.2088	24.0	26.1534	.3061		27.0710
$\zeta = 1.02; \omega_{n1} = 16.0$					$\zeta = 1.50; \omega_{n1} = 1.0$					$\zeta = 1.50; \omega_{n1} = 8.0$				
.1	.1829	.7264	-2	.1334	.1	.1839	-.7667	-2	.8388	.1	.1772	.4930	-2	.6223
.2	.3396	.2472	-1	.4560	.2	.3462	-.2468	-1	.2915	.2	.3242	.1601	-1	.2038
.3	.4765	.4825	-1	.0935	.3	.4956	-.4612	-1	.5833	.3	.4540	.3025	-1	.3880
.4	.6049	.7550	-1	1.4031	.4	.6367	-.6958	-1	.9374	.4	.5730	.4622	-1	.5968
.5	.7222	.1050		1.9580	.5	.7722	-.9375	-1	1.341	.5	.6848	.6313	-1	.8202
.6	.8330	.1359		2.5410	.6	.9037	-.1179		1.784	.6	.7918	.8052	-1	1.0523
.7	.9387	.1674		3.1400	.7	1.0322	-.1415		.2262	.7	.8955	.9808	-1	1.2892
.8	1.0408	.1992		3.7454	.8	1.1586	-.1644		.2770	.8	.9967	.1156		1.5282
.9	1.1400	.2309		4.3540	.9	1.2832	-.1863		.3305	.9	1.0963	.1330		1.7675
1.0	1.2372	.2622		4.9580	1.0	1.4065	-.2071		.3864	1.0	1.1947	.1501		2.0059
1.5	1.7063	.4082		7.8243	1.5	2.0090	-.2951		.6959	1.5	1.6784	.2298		3.1597
2.0	2.1678	.5300		10.3114	2.0	2.5999	-.3590		1.0451	2.0	2.1597	.2983		4.2268
2.5	2.6332	.6258		12.3804	2.5	3.1707	-.4047		1.4246	2.5	2.6295	.3674		5.2942
3.0	3.1053	.6984		14.0687	3.0	3.7348	-.4375		1.8276	3.0	3.1295	.4042		6.3604
3.5	3.5841	.7516		15.4417	3.5	4.2950	-.4691		2.2659	3.5	3.6095	.4476		7.4150
4.0	4.0686	.7898		16.5686	4.0	4.8552	-.5005		2.6893	4.0	4.0874	.4868		8.4646
5.0	5.0495	.8355		18.3185	5.0	5.4154	-.5302		3.0995	5.0	4.5688	.5268		9.5150
6.0	6.0402	.8752		19.6754	6.0	5.9756	-.5591		3.4995	6.0	5.0474	.5668		10.5650
7.0	7.0359	.9073		20.8403	7.0	6.5358	-.5875		3.8995	7.0	5.5268	.6068		11.6150
8.0	8.0339	.9378		21.9152	8.0	7.0960	-.6154		4.2995	8.0	6.0074	.6468		12.6650
9.0	9.0330	.9673		22.9489	9.0	7.6562	-.6429		4.6995	9.0	6.4874	.6868		13.7150
10.0	10.0326	.9948		23.9638	10.0	8.2164	-.6700		5.0995	10.0	6.9674	.7268		14.7650
11.0	11.0324	.9792		24.9705	11.0	8.7766	-.6967		5.4995	11.0	7.4474	.7668		15.8150
12.0	12.0323	.9754		25.9735	12.0	9.3368	-.7230		5.8995	12.0	7.9274	.8068		16.8650
13.0	13.0323	.9754		26.9746	13.0	9.8970	-.7489		6.2995	13.0	8.4074	.8468		17.9150
14.0	14.0323	.9755		27.9755	14.0	10.4572	-.7744		6.6995	14.0	8.8874	.8868		18.9650
15.0	15.0323	.9755		28.9755	15.0	11.0174	-.7995		7.0995	15.0	9.3674	.9268		19.9650
$\zeta = 1.50; \omega_{n1} = 0$					$\zeta = 1.50; \omega_{n1} = 2.0$					$\zeta = 1.50; \omega_{n1} = 16.0$				
.5	1.1321	1.3279		.6379	.1	.1801	-.1922	-2	.1607	.1	.1767	.6283	-2	.1238
1.0	1.9812	2.2168		.2944	.2	.3335	-.6230	-2	.5399	.2	.3227	.2041	-1	.4038
1.5	2.7239	2.9731		.5252	.3	.4715	-.1174	-1	.1050	.3	.4511	.3856	-1	.7658
2.0	3.4041	3.6992		.8909	.4	.5996	-.1789	-1	.1646	.4	.5687	.5092	-1	1.1740
2.5	4.0426	4.3008		1.2152	.5	.7210	-.2436	-1	.2303	.5	.6791	.6050	-1	1.6088
3.0	4.6507	4.9103		1.6088	.6	.8378	-.3097	-1	.3003	.6	.7846	.7027	-1	2.0583
3.5	5.2377	5.4979		2.0237	.7	.9511	-.3761	-1	.3737	.7	.8869	.8000	-1	2.5147
4.0	5.8058	6.0672		2.4556	.8	1.0619	-.4420	-1	.4497	.8	.9867	.8967	-1	2.9729
4.5	6.3611	6.6229		2.9008	.9	1.1709	-.5068	-1	.5277	.9	1.0850	.9867	-1	3.4292
5.0	6.9034	7.1656		3.3587	1.0	1.2785	-.5703	-1	.6075	1.0	1.1822	.1915		3.8814
5.5	7.4398	7.7022		3.8225	1.5	1.8040	-.8613	-1	1.0238	1.5	1.6606	.2933		6.0321
6.0	7.9718	8.2342		4.2907	2.0	2.3199	-.1105		1.4590	2.0	2.1379	.3809		7.9570
6.5	8.4967	8.7594		4.7660	2.5	2.8318	-.1307		1.9068	2.5	2.6184	.4548		9.6578
7.0	9.0163	9.2791		5.2464	3.0	3.3414	-.1472		2.3641	3.0	3.1025	.5165		11.1608
					4.0	4.3560	-.1719		3.3002	3.5	3.5896	.5678		12.4949
					5.0	5.3661	-.1886		4.2567	4.0	4.0792	.6104		13.6875
					6.0	6.3731	-.1998		5.2272	4.5	4.5640	.6514		15.7379
					8.0	8.3813	-.2127		7.1934					

TABLE II.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 1 THIRD-ORDER PLANT  
WITH ZERO - Concluded

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0 \omega_n / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_m$	b/B	$R_0 \omega_n / BK$
$\zeta = 2.00; \omega_n T_1 = 0$					$\zeta = 2.00; \omega_n T_1 = 1.0$					$\zeta = 2.00; \omega_n T_1 = 4.0$				
0.5	1.1032	1.2615		0.5509 -1	6.0		7.3185	-0.7214	3.9602	20.0		20.1688	0.6072 -1	20.0740
1.0	1.9773	2.1560		.2014	8.0		9.4817	-.7262	5.7921	22.0		22.1688	.6085 -1	22.0746
1.5	2.7624	2.9456		.4209	10.0		11.5750	-.7280	7.6970	24.0		24.1688	.6094 -1	24.0750
2.0	3.4921	3.6769		.6927	12.0		13.6288	-.7287	9.6425	26.0		26.1688	.6094 -1	26.0750
2.5	4.1809	4.3661		1.0042	14.0		15.6600	-.7291	11.6109	28.0		28.1688	.6094 -1	28.0750
3.0	4.8382	5.0238		1.3473	16.0		17.6781	-.7293	13.5926	30.0		30.1687	.6103 -1	30.0754
3.5	5.4708	5.6562		1.7147	18.0		19.6887	-.7294	15.5819	$\zeta = 2.00; \omega_n T_1 = 8.0$				
4.0	6.0842	6.2698		2.1014	20.0		21.6949	-.7294	17.5757	.1		.1716	.3695 -2	.5792 -1
4.5	6.6787	6.8643		2.5068	22.0		23.6985	-.7295	19.5720	.2		.3101	.1152 -1	.1821
5.0	7.2567	7.4423		2.9288	24.0		25.7006	-.7295	21.5699	.3		.4322	.2121 -1	.3375
5.5	7.8263	8.0119		3.3593	26.0		27.7019	-.7295	23.5686	.4		.5451	.3181 -1	.5094
6.0	8.3854	8.5710		3.8003	28.0		29.7025	-.7295	25.5679	.5		.6525	.4285 -1	.6903
6.5	8.9373	9.1230		4.2485	30.0		31.7030	-.7295	27.5675	.6		.7565	.5407 -1	.8761
7.0	9.4822	9.6679		4.7036	$\zeta = 2.00; \omega_n T_1 = 2.0$					.7		.8582	.6531 -1	1.0643
7.5	10.0213	10.2071		5.1645	.1		.1743	-.5370 -2	.1494 -1	.8		.9585	.7647 -1	1.2533
8.0	10.5540	10.7397		5.6316	.2		.3185	-.1620 -1	.4810 -1	.9		1.0578	.8749 -1	1.4421
8.5	11.0834	11.2691		6.1023	.3		.4477	-.3070 -1	.9094 -1	1.0		1.1565	.9833 -1	1.6301
9.0	11.6086	11.7943		6.5770	.4		.5684	-.4593 -1	.1398	1.5		1.6468	.1492	2.5467
9.5	12.1305	12.3161		7.0551	.5		.6840	-.6171 -1	.1926	2.0		2.1366	.1942	3.4170
10.0	12.6493	12.8350		7.5364	.6		.7962	-.7767 -1	.2484	3.0		3.1200	.2685	5.0278
$\zeta = 2.00; \omega_n T_1 = 0.5$					.7		.9063	-.9356 -1	.3065	4.0		4.1079	.3256	6.4970
.1		.1852	-0.2119 -1	.4229 -2	.8		1.0150	-.1093	.3665	6.0		6.0923	.4031	9.1328
.2		.3530	-.6405	.1495 -1	.9		1.1227	-.1247	.4280	8.0		8.0836	.4487	11.5059
.3		.5129	-.1132	.3052 -1	1.0		1.2296	-.1397	.4909	10.0		10.0785	.4754	13.7249
.4		.6689	-.1620	.5013 -1	1.5		1.7593	-.2093	.8222	12.0		12.0756	.4911	15.8530
.5		.8228	-.2075	.7331 -1	2.0		2.2854	-.2691	1.1765	14.0		14.0740	.5002	17.9280
.6		.9761	-.2482	.9975 -1	3.0		3.3318	-.3640	1.9401	16.0		16.0730	.5056	19.9718
.7		1.1289	-.2837	.1292	4.0		4.3717	-.4334	2.7614	18.0		18.0724	.5088	21.9976
.8		1.2814	-.3140	.1616	6.0		6.4330	-.5218	4.5234	20.0		20.0721	.5106	24.0125
.9		1.4335	-.3396	.1967	8.0		8.4735	-.5702	6.3860	22.0		22.0719	.5116	26.0213
1.0		1.5852	-.3608	.2344	10.0		10.4992	-.5972	8.3064	24.0		24.0718	.5123	28.0268
1.5		2.3308	-.4224	.4580	12.0		12.5149	-.6125	10.2600	26.0		26.0717	.5127	30.0296
2.0		3.0464	-.4454	.7309	14.0		14.5244	-.6213	12.2330	28.0		28.0717	.5128	32.0307
3.0		4.3854	-.4588	1.3852	16.0		16.5300	-.6264	14.2172	30.0		30.0717	.5129	34.0318
4.0		5.6285	-.4620	2.1405	18.0		18.5333	-.6294	16.2080	$\zeta = 2.00; \omega_n T_1 = 16.0$				
6.0		7.9345	-.4634	3.8338	20.0		20.5353	-.6311	18.2026	.1		.1712	.5402 -2	.1153
8.0		10.1029	-.4637	5.6653	22.0		22.5364	-.6321	20.1994	.2		.3087	.1685 -1	.3610
10.0		12.1980	-.4638	7.5701	24.0		24.5371	-.6327	22.1976	.3		.4297	.3102 -1	.6666
12.0		14.2525	-.4639	9.5155	26.0		26.5375	-.6330	24.1965	.4		.5414	.4652 -1	1.0030
14.0		16.2841	-.4639	11.4839	28.0		28.5377	-.6332	26.1959	.5		.6476	.6268 -1	1.3553
16.0		18.3025	-.4639	13.4656	30.0		30.5379	-.6333	28.1955	.6		.7503	.7910 -1	1.7153
18.0		20.3132	-.4639	15.4549	$\zeta = 2.00; \omega_n T_1 = 4.0$					.7		.8509	.9556 -1	2.0781
20.0		22.3194	-.4639	17.4486	.1		.1725	.4478 -2	.2926 -1	.8		.9500	.1119	2.4404
22.0		24.3230	-.4639	19.4450	.2		.3128	.1396 -2	.9275 -1	.9		1.0482	.1280	2.8005
24.0		26.3252	-.4639	21.4428	.3		.4373	.2567 -2	.1730	1.0		1.1459	.1439	3.1568
26.0		28.3264	-.4639	23.4416	.4		.5527	.3850 -2	.2627	1.5		1.6315	.2184	4.8635
28.0		30.3272	-.4639	25.4409	.5		.6627	.5183 -2	.3581	2.0		2.1176	.2844	6.4336
30.0		32.3276	-.4639	27.4404	.6		.7692	.6537 -2	.4570	2.5		2.6056	.3425	7.8743
$\zeta = 2.00; \omega_n T_1 = 1.0$					.7		.8735	.7892 -2	.5581	3.0		3.0955	.3935	9.1998
.1		.1779	-.1428 -1	.7786 -2	.8		.9762	.9236 -2	.6607	4.0		4.0797	.4774	11.5582
.2		.3299	-.4420 -1	.2587 -1	.9		1.0780	.1056 -1	.7642	5.0		5.0681	.5418	13.6009
.3		.4692	-.8055 -1	.5022 -1	1.0		1.1791	.1187 -1	.8684	6.0		6.0596	.5913	15.4007
.4		.6015	-.1195	.7901 -1	1.5		1.6801	.1797 -1	1.3917	8.0		8.0484	.6582	18.4833
.5		.7298	-.1590	.1113	2.0		2.1790	.2335 -1	1.9144	10.0		10.0421	.6975	21.1187
.6		.8556	-.1980	.1464	3.0		3.1766	.3219 -1	2.9521	12.0		12.0384	.7206	23.4905
.7		.9800	-.2360	.1840	4.0		4.1748	.3897 -1	3.9811	14.0		14.0363	.7340	25.7077
.8		1.1036	-.2725	.2239	6.0		6.1723	.4812 -1	5.0202	16.0		16.0351	.7419	27.8357
.9		1.2268	-.3072	.2659	8.0		8.1708	.5348 -1	6.0431	18.0		18.0344	.7466	29.9107
1.0		1.3499	-.3402	.3099	10.0		10.1699	.5663 -1	7.0566	20.0		20.0339	.7492	31.9540
1.5		1.9671	-.4764	.5565	12.0		12.1694	.5846 -1	8.0644	22.0		22.0337	.7508	33.9795
2.0		2.5887	-.5694	.8420	14.0		14.1692	.5954 -1	9.0690	24.0		24.0336	.7518	35.9948
3.0		3.8279	-.6654	1.5067	16.0		16.1690	.6017 -1	10.0717	26.0		26.0335	.7523	38.0038
4.0		5.0336	-.7011	2.2653	18.0		18.1689	.6054 -1	11.0733	28.0		28.0334	.7526	40.0089
										30.0		30.0334	.7528	42.0114

TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER  
UNSTABLE PLANT WITH ZERO

$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	b/B	$R_0/BK\tau_2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	b/B	$R_0/BK\tau_2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	b/B	$R_0/BK\tau_2$
$\tau_2/\tau_3 = 0.10; \tau_1/\tau_2 = 0$					$\tau_2/\tau_3 = 0.10; \tau_1/\tau_2 = 2.0$					$\tau_2/\tau_3 = 0.10; \tau_1/\tau_2 = 5.0$				
0.2528	-1	0.7528	-1	0.0281	-1	0.5048	-1	0.1	-0.7314	-3	0.5002	-3	0.5086	-1
.5112	-1	.1511	.2	.2529	-4	.1019	.2	.2	-.2850	-2	.1997	-2	.1034	.2
.7751	-1	.2275	.3	.8445	-4	.1540	.3	.3	-.6247	-2	.4485	-2	.1575	.3
.1044	.3045	.4	.1996	-3	.2068	.4	.1681	.4	-.1681	-1	.7956	-2	.2130	.4
.1319	.3821	.5	.3905	-3	.2602	.5	-.1644	.5	-.1644	-1	.1240	-1	.2699	.5
.1599	.4602	.6	.6730	-3	.3141	.6	-.2304	.6	-.2304	-1	.1779	-1	.3280	.6
.1884	.5390	.7	.1067	-2	.3684	.7	-.3050	.7	-.3050	-1	.2413	-1	.3873	.7
.2175	.6183	.8	.1590	-2	.4230	.8	-.3873	.8	-.3873	-1	.3139	-1	.4476	.8
.2470	.6981	.9	.2261	-2	.4779	.9	-.4764	.9	-.4764	-1	.3955	-1	.5089	.9
.2770	.7785	1.0	.3095	-2	.5329	1.0	-.5715	1.0	-.5715	-1	.4899	-1	.5710	1.0
.3070	.8589	1.1	.4032	-1	.5879	1.1	-.6809	1.1	-.6809	-1	.5907	-1	.6361	1.1
.3370	.9393	1.2	.5070	-1	.6429	1.2	-.7956	1.2	-.7956	-1	.7056	-1	.7056	1.2
.3670	1.0197	1.3	.6208	-1	.6979	1.3	-.9104	1.3	-.9104	-1	.8204	-1	.8204	1.3
.3970	1.0997	1.4	.7346	-1	.7529	1.4	-.1.0251	1.4	-.1.0251	-1	.9351	-1	.9351	1.4
.4270	1.1797	1.5	.8484	-1	.8079	1.5	-.1.1399	1.5	-.1.1399	-1	1.0499	-1	1.0499	1.5
.4570	1.2597	1.6	.9622	-1	.8629	1.6	-.1.2547	1.6	-.1.2547	-1	1.1647	-1	1.1647	1.6
.4870	1.3397	1.7	1.0760	-1	.9179	1.7	-.1.3695	1.7	-.1.3695	-1	1.2795	-1	1.2795	1.7
.5170	1.4197	1.8	1.1898	-1	.9729	1.8	-.1.4843	1.8	-.1.4843	-1	1.3943	-1	1.3943	1.8
.5470	1.4997	1.9	1.3036	-1	1.0279	1.9	-.1.5991	1.9	-.1.5991	-1	1.5091	-1	1.5091	1.9
.5770	1.5797	2.0	1.4174	-1	1.0829	2.0	-.1.7139	2.0	-.1.7139	-1	1.6239	-1	1.6239	2.0
.6070	1.6597	2.1	1.5312	-1	1.1379	2.1	-.1.8287	2.1	-.1.8287	-1	1.7387	-1	1.7387	2.1
.6370	1.7397	2.2	1.6450	-1	1.1929	2.2	-.1.9435	2.2	-.1.9435	-1	1.8535	-1	1.8535	2.2
.6670	1.8197	2.3	1.7588	-1	1.2479	2.3	-.1.0583	2.3	-.1.0583	-1	1.9683	-1	1.9683	2.3
.6970	1.8997	2.4	1.8726	-1	1.3029	2.4	-.1.1731	2.4	-.1.1731	-1	2.0831	-1	2.0831	2.4
.7270	1.9797	2.5	1.9864	-1	1.3579	2.5	-.1.2879	2.5	-.1.2879	-1	2.1979	-1	2.1979	2.5
.7570	2.0597	2.6	2.0998	-1	1.4129	2.6	-.1.4027	2.6	-.1.4027	-1	2.3127	-1	2.3127	2.6
.7870	2.1397	2.7	2.2136	-1	1.4679	2.7	-.1.5175	2.7	-.1.5175	-1	2.4275	-1	2.4275	2.7
.8170	2.2197	2.8	2.3274	-1	1.5229	2.8	-.1.6323	2.8	-.1.6323	-1	2.5423	-1	2.5423	2.8
.8470	2.2997	2.9	2.4412	-1	1.5779	2.9	-.1.7471	2.9	-.1.7471	-1	2.6571	-1	2.6571	2.9
.8770	2.3797	3.0	2.5550	-1	1.6329	3.0	-.1.8619	3.0	-.1.8619	-1	2.7719	-1	2.7719	3.0
.9070	2.4597	3.1	2.6688	-1	1.6879	3.1	-.1.9767	3.1	-.1.9767	-1	2.8867	-1	2.8867	3.1
.9370	2.5397	3.2	2.7826	-1	1.7429	3.2	-.2.0915	3.2	-.2.0915	-1	2.9995	-1	2.9995	3.2
.9670	2.6197	3.3	2.8964	-1	1.7979	3.3	-.2.2063	3.3	-.2.2063	-1	3.1143	-1	3.1143	3.3
.9970	2.6997	3.4	3.0098	-1	1.8529	3.4	-.2.3211	3.4	-.2.3211	-1	3.2291	-1	3.2291	3.4
1.0270	2.7797	3.5	3.1236	-1	1.9079	3.5	-.2.4359	3.5	-.2.4359	-1	3.3439	-1	3.3439	3.5
1.0570	2.8597	3.6	3.2374	-1	1.9629	3.6	-.2.5507	3.6	-.2.5507	-1	3.4587	-1	3.4587	3.6
1.0870	2.9397	3.7	3.3512	-1	2.0179	3.7	-.2.6655	3.7	-.2.6655	-1	3.5735	-1	3.5735	3.7
1.1170	3.0197	3.8	3.4650	-1	2.0729	3.8	-.2.7803	3.8	-.2.7803	-1	3.6883	-1	3.6883	3.8
1.1470	3.0997	3.9	3.5788	-1	2.1279	3.9	-.2.8951	3.9	-.2.8951	-1	3.8031	-1	3.8031	3.9
1.1770	3.1797	4.0	3.6926	-1	2.1829	4.0	-.3.0099	4.0	-.3.0099	-1	3.9179	-1	3.9179	4.0
1.2070	3.2597	4.1	3.8064	-1	2.2379	4.1	-.3.1247	4.1	-.3.1247	-1	4.0327	-1	4.0327	4.1
1.2370	3.3397	4.2	3.9202	-1	2.2929	4.2	-.3.2395	4.2	-.3.2395	-1	4.1475	-1	4.1475	4.2
1.2670	3.4197	4.3	4.0340	-1	2.3479	4.3	-.3.3543	4.3	-.3.3543	-1	4.2623	-1	4.2623	4.3
1.2970	3.4997	4.4	4.1478	-1	2.4029	4.4	-.3.4691	4.4	-.3.4691	-1	4.3771	-1	4.3771	4.4
1.3270	3.5797	4.5	4.2616	-1	2.4579	4.5	-.3.5839	4.5	-.3.5839	-1	4.4919	-1	4.4919	4.5
1.3570	3.6597	4.6	4.3754	-1	2.5129	4.6	-.3.6987	4.6	-.3.6987	-1	4.6067	-1	4.6067	4.6
1.3870	3.7397	4.7	4.4892	-1	2.5679	4.7	-.3.8135	4.7	-.3.8135	-1	4.7215	-1	4.7215	4.7
1.4170	3.8197	4.8	4.6030	-1	2.6229	4.8	-.3.9283	4.8	-.3.9283	-1	4.8363	-1	4.8363	4.8
1.4470	3.8997	4.9	4.7168	-1	2.6779	4.9	-.4.0431	4.9	-.4.0431	-1	4.9511	-1	4.9511	4.9
1.4770	3.9797	5.0	4.8306	-1	2.7329	5.0	-.4.1579	5.0	-.4.1579	-1	5.0659	-1	5.0659	5.0
1.5070	4.0597	5.1	4.9444	-1	2.7879	5.1	-.4.2727	5.1	-.4.2727	-1	5.1807	-1	5.1807	5.1
1.5370	4.1397	5.2	5.0582	-1	2.8429	5.2	-.4.3875	5.2	-.4.3875	-1	5.2955	-1	5.2955	5.2
1.5670	4.2197	5.3	5.1720	-1	2.8979	5.3	-.4.5023	5.3	-.4.5023	-1	5.4103	-1	5.4103	5.3
1.5970	4.2997	5.4	5.2858	-1	2.9529	5.4	-.4.6171	5.4	-.4.6171	-1	5.5251	-1	5.5251	5.4
1.6270	4.3797	5.5	5.3996	-1	3.0079	5.5	-.4.7319	5.5	-.4.7319	-1	5.6399	-1	5.6399	5.5
1.6570	4.4597	5.6	5.5134	-1	3.0629	5.6	-.4.8467	5.6	-.4.8467	-1	5.7547	-1	5.7547	5.6
1.6870	4.5397	5.7	5.6272	-1	3.1179	5.7	-.4.9615	5.7	-.4.9615	-1	5.8695	-1	5.8695	5.7
1.7170	4.6197	5.8	5.7410	-1	3.1729	5.8	-.5.0763	5.8	-.5.0763	-1	5.9843	-1	5.9843	5.8
1.7470	4.6997	5.9	5.8548	-1	3.2279	5.9	-.5.1911	5.9	-.5.1911	-1	6.0991	-1	6.0991	5.9
1.7770	4.7797	6.0	5.9686	-1	3.2829	6.0	-.5.3059	6.0	-.5.3059	-1	6.2139	-1	6.2139	6.0
1.8070	4.8597	6.1	6.0824	-1	3.3379	6.1	-.5.4207	6.1	-.5.4207	-1	6.3287	-1	6.3287	6.1
1.8370	4.9397	6.2	6.1962	-1	3.3929	6.2	-.5.5355	6.2	-.5.5355	-1	6.4435	-1	6.4435	6.2
1.8670	5.0197	6.3	6.3098	-1	3.4479	6.3	-.5.6503	6.3	-.5.6503	-1	6.5583	-1	6.5583	6.3
1.8970	5.0997	6.4	6.4236	-1	3.5029	6.4	-.5.7651	6.4	-.5.7651	-1	6.6731	-1	6.6731	6.4
1.9270	5.1797	6.5	6.5374	-1	3.5579	6.5	-.5.8799	6.5	-.5.8799	-1	6.7879	-1	6.7879	6.5
1.9570	5.2597	6.6	6.6512	-1	3.6129	6.6	-.5.9947	6.6	-.5.9947	-1	6.9027	-1	6.9027	6.6
1.9870	5.3397	6.7	6.7650	-1	3.6679	6.7	-.6.1095	6.7	-.6.1095	-1	7.0175	-1	7.0175	6.7
2.0170	5.4197	6.8	6.8788	-1	3.7229	6.8	-.6.2243	6.8	-.6.2243	-1	7.1323	-1	7.1323	6.8
2.0470	5.4997	6.9	6.9926	-1	3.7779	6.9	-.6.3391	6.9	-.6.3391	-1	7.2471	-1	7.2471	6.9
2.0770	5.5797	7.0	7.1064	-1	3.8329	7.0	-.6.4539	7.0	-.6.4539	-1	7.3619	-1	7.3619	7.0
2.1070	5.6597	7.1	7.2202	-1	3.8879	7.1	-.6.5687	7.1	-.6.5687	-1	7.4767	-1	7.4767	7.1
2.1370	5.7397	7.2	7.3340	-1	3.9429	7.2	-.6.6835	7.2	-.6.6835	-1	7.5915	-1	7.5915	7.2
2.1670	5.8197	7.3	7.4478	-1	4.0079	7.3	-.6.7983	7.3	-.6.7983	-1	7.7063	-1	7.7063	7.3
2.1970	5.8997	7.4	7.5616	-1	4.0629	7.4	-.6.9131	7.4	-.6.9131	-1	7.8211	-1	7.8211	7.4
2.2270	5.9797	7.5	7.6754	-1	4.1179	7.5	-.7.0279	7.5	-.7.0279	-1	7.9359	-1	7.9359	7.5
2.2570	6.0597	7.6	7.7892	-1	4.1729	7.6	-.7.1427	7.6	-.7.1427	-1	8.0507	-1	8.0507	7.6
2.2870	6.1397	7.7	7.9030	-1	4.2279	7.7	-.7.2575	7.7	-.7.2575	-1	8.1655	-1	8.1655	7.7
2.3170	6.2197	7.8	8.0168	-1	4.2829	7.8	-.7.3723	7.8	-.7.3723	-1	8.2803	-1	8.2803	7.8
2.3470	6.2997	7.9	8.1306	-1	4.3379	7.9	-.7.4871	7.9	-.7.4871	-1	8.3951	-1	8.3951	7.9
2.3770	6.3797	8.0	8.2444	-1	4.3929	8.0	-.7.6019	8.0	-.7.6019	-1	8.5099	-1	8.5099	8.0
2.4070	6.4597	8.1	8.3582	-1	4.4479	8.1	-.7.7167	8.1	-.7.7167	-1	8.6247	-1	8.6247	8.1
2.4370	6.5397	8.2	8.4720	-1	4.5029	8.2	-.7.8315	8.2	-.7.8315	-1	8.7395	-1	8.7395	8.2
2.4670	6.6197	8.3	8.5858	-1	4.5579	8.3	-.7.9463	8.3	-.7.9463	-1	8.8543	-1	8.8543	8.3
2.4970	6.6997	8.4												



TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER  
UNSTABLE PLANT WITH ZERO - Continued

$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2$
$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 0.5$					$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 3.0$					$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 0$				
0.4857 -1	0.1	0.5099 -2	0.3128 -3	0.5050 -1	0.1	0.9556 -3	0.1873 -2	0.2515 -1	0.7516 -1	0.1	0.5099 -1	1.507	0.2	0.1562 -4
0.9473 -1	0.2	0.1857 -1	0.1258 -2	0.1019	0.2	0.3755 -2	0.7486 -2	0.5059 -1	1.507	0.2	0.5099 -1	1.507	0.2	0.1249 -3
1.1390	0.3	0.3818 -1	0.2849 -2	0.1540	0.3	0.8293 -2	1.681 -1	0.7630	2.265	0.3	0.7630	2.265	0.3	0.4215 -3
1.1819	0.4	0.6226 -1	0.5111 -2	0.2068	0.4	0.1446 -1	0.2979 -1	1.022	3.027	0.4	1.022	3.027	0.4	0.9983 -3
2.236	0.5	0.8954	0.8069 -2	0.2600	0.5	0.2215 -1	0.4638 -1	1.284	3.794	0.5	1.284	3.794	0.5	0.1948 -2
2.644	0.6	1.191	0.1175 -1	0.3136	0.6	0.3124 -1	0.6651 -1	1.547	4.564	0.6	1.547	4.564	0.6	0.3362 -2
3.044	0.7	1.500	0.1620 -1	0.3674	0.7	0.4164 -1	0.9008 -1	1.812	5.339	0.7	1.812	5.339	0.7	0.5332 -2
3.438	0.8	1.820	0.2143 -1	0.4213	0.8	0.5321 -1	1.170	2.079	6.118	0.8	2.079	6.118	0.8	0.7946 -2
3.827	0.9	2.144	0.2749 -1	0.4752	0.9	0.6586 -1	1.472	2.346	6.902	0.9	2.346	6.902	0.9	0.1129 -1
4.211	1.0	2.469	0.3440 -1	0.5289	1.0	0.7947 -1	1.806	2.614	7.691	1.0	2.614	7.691	1.0	0.1546 -1
4.607	1.5	4.037	0.8280 -1	0.7923	1.5	0.1584	3.907	3.955	1.1712	1.5	3.955	1.1712	1.5	0.5148 -1
4.785	2.0	5.410	0.1565	1.0393	2.0	0.2467	6.617	5.270	1.5869	2.0	5.270	1.5869	2.0	0.1197
4.977	2.5	6.543	0.2575	1.2628	2.5	0.3344	9.776	6.525	2.0165	2.5	6.525	2.0165	2.5	0.2276
1.1203	3.0	7.445	0.3871	1.4596	3.0	0.4141	1.3232	7.690	2.4596	3.0	7.690	2.4596	3.0	0.3812
1.2737	3.5	8.143	0.5454	1.6293	3.5	0.4814	1.6857	8.742	2.9153	3.5	8.742	2.9153	3.5	0.5822
1.4172	4.0	8.672	0.7320	1.7741	4.0	0.5349	2.0566	9.668	3.3820	4.0	9.668	3.3820	4.0	0.8303
1.5505	4.5	9.064	0.9459	1.8973	4.5	0.5773	2.4313	1.0464	3.8577	4.5	1.0464	3.8577	4.5	1.1226
1.6734	5.0	9.349	1.1858	2.0026	5.0	0.6046	2.8086	1.1134	4.3406	5.0	1.1134	4.3406	5.0	1.4544
1.7862	5.5	9.553	1.4500	2.0933	5.5	0.6252	3.1890	1.1687	4.8288	5.5	1.1687	4.8288	5.5	1.8202
1.8891	6.0	9.697	1.7370	2.1720	6.0	0.6393	3.5739	1.2139	5.3210	6.0	1.2139	5.3210	6.0	2.2142
1.9826	6.5	9.797	2.0449	2.2409	6.5	0.6488	3.9646	1.2502	5.8158	6.5	1.2502	5.8158	6.5	2.6312
2.0673	7.0	9.865	2.3721	2.3015	7.0	0.6551	4.3624	1.2793	6.3125	7.0	1.2793	6.3125	7.0	3.0664
2.1437	7.5	9.911	2.7170	2.3551	7.5	0.6593	4.7677	1.3023	6.8104	7.5	1.3023	6.8104	7.5	3.5161
2.2125	8.0	9.942	3.0778	2.4025	8.0	0.6619	5.1809	1.3205	7.3090	8.0	1.3205	7.3090	8.0	3.9770
2.2743	8.5	9.962	3.4532	2.4446	8.5	0.6637	5.6019	1.3349	7.8082	8.5	1.3349	7.8082	8.5	4.4467
2.3297	9.0	9.976	3.8418	2.4819	9.0	0.6648	6.0305	1.3461	8.3077	9.0	1.3461	8.3077	9.0	4.9231
2.3792	9.5	9.985	4.2423	2.5151	9.5	0.6655	6.4662	1.3549	8.8074	9.5	1.3549	8.8074	9.5	5.4049
2.4235	10.0	9.990	4.6535	2.5446	10.0	0.6659	6.9086	1.3618	9.3072	10.0	1.3618	9.3072	10.0	5.8907
2.4681	11.0	9.996	5.5040	2.5940	11.0	0.6664	7.8112	1.3714	10.3070	11.0	1.3714	10.3070	11.0	6.3711
2.5172	12.0	9.998	6.3856	2.6328	12.0	0.6666	8.7341	1.3773	11.3070	12.0	1.3773	11.3070	12.0	7.8593
2.5646	14.0	1.0000	8.2188	2.6872	14.0	0.6667	10.6255	1.3808	12.3070	13.0	1.3808	12.3070	13.0	8.8521
2.6200	16.0	1.0000	10.1160	2.7206	16.0	0.6667	12.5588	1.3830	13.3069	14.0	1.3830	13.3069	14.0	9.8478
2.7235	18.0	1.0000	12.0530	2.7410	18.0	0.6667	14.5180	1.3843	14.3069	15.0	1.3843	14.3069	15.0	10.8452
2.7428	20.0	1.0000	14.0145	2.7534	20.0	0.6667	16.4932	1.3851	15.3069	16.0	1.3851	15.3069	16.0	11.8436
2.7545	22.0	1.0000	15.9911	2.7609	22.0	0.6667	18.4781	1.3856	16.3069	17.0	1.3856	16.3069	17.0	12.8426
2.7616	24.0	1.0000	17.9768	2.7655	24.0	0.6667	20.4690	1.3858	17.3069	18.0	1.3858	17.3069	18.0	13.8420
2.7659	26.0	1.0000	19.9682	2.7683	26.0	0.6667	22.4634	1.3860	18.3069	19.0	1.3860	18.3069	19.0	14.8417
2.7685	28.0	1.0000	21.9629	2.7700	28.0	0.6667	24.4600	1.3861	19.3069	20.0	1.3861	19.3069	20.0	15.8414
2.7701	30.0	1.0000	23.9597	2.7710	30.0	0.6667	26.4580	1.3862	20.3069	21.0	1.3862	20.3069	21.0	16.8413
2.7711	32.0	1.0000	25.9578	2.7716	32.0	0.6667	28.4567							
2.7717	34.0	1.0000	27.9566	2.7720	34.0	0.6667	30.4560							
2.7720	36.0	1.0000	29.9559	2.7722	36.0	0.6667	32.4555							
2.7723	38.0	1.0000	31.9555	2.7724	38.0	0.6667	34.4552							
2.7724	40.0	1.0000	33.9552	2.7725	40.0	0.6667	36.4551							
2.7725	42.0	1.0000	35.9551											
$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 2.0$					$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 5.0$					$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 0.5$				
0.5029 -1	0.1	0.9140 -3	0.1249 -2	0.5066 -1	0.1	0.8905 -3	0.3123 -2	0.4827 -1	0.1	0.9666 -2	0.6259 -3			
1.0111	0.2	0.3562 -2	0.4992 -2	0.1026	0.2	0.3522 -2	1.248 -1	0.9359 -1	0.2	0.2063 -1	0.2515 -2			
1.1522	0.3	0.7805 -2	1.1121 -1	0.1556	0.3	0.7829 -2	2.801 -1	1.362	0.3	0.4240 -1	0.5699 -2			
2.036	0.4	0.1351 -1	0.1988 -1	0.2053	0.4	0.1374 -1	0.4963 -1	1.776	0.4	0.6911 -1	1.022 -1			
2.2550	0.5	0.2053 -1	0.3097 -1	0.2643	0.5	0.2118 -1	0.7725 -1	2.171	0.5	0.9935 -1	1.614 -1			
2.3065	0.6	0.2874 -1	0.4443 -1	0.3198	0.6	0.3006 -1	1.1107	2.552	0.6	1.320	0.2351 -1			
3.579	0.7	0.3801 -1	0.6022 -1	0.3758	0.7	0.4031 -1	1.499	2.922	0.7	1.663	0.3240 -1			
4.091	0.8	0.4823 -1	0.7828 -1	0.4322	0.8	0.5183 -1	1.947	3.282	0.8	2.016	0.4287 -1			
4.600	0.9	0.5926 -1	0.9857 -1	0.4889	0.9	0.6454 -1	2.449	3.632	0.9	2.374	0.5498 -1			
5.105	1.0	0.7101	1.210	0.5458	1.0	0.7836 -1	3.002	3.973	1.0	2.732	0.6879 -1			
5.515	1.5	1.1368	0.2634	0.8085	1.5	0.1610	6.478	4.267	1.5	4.448	1.651			
5.916	2.0	0.2065	0.4498	1.0991	2.0	0.2583	1.0934	4.563	2.0	5.919	3.099			
1.1864	2.5	0.2722	0.6717	1.3473	2.5	0.3604	1.6076	4.820	2.5	7.092	5.049			
1.3685	3.0	0.3293	0.9215	1.5661	3.0	0.4584	2.1600	5.054	3.0	8.264	6.747			
1.5290	3.5	0.3758	1.1935	1.7524	3.5	0.5455	2.7226	5.260	3.5	9.409	8.326			
1.6700	4.0	0.4119	1.4837	1.9067	4.0	0.6174	3.2737	5.438	4.0	1.054	1.0375			
1.7940	4.5	0.4388	1.7895	2.0325	4.5	0.6732	3.8009	5.595	4.5	1.153	1.3662			
1.9033	5.0	0.4582	2.1098	2.1349	5.0	0.7142	4.3012	5.739	5.0	1.239	1.7289			
2.0000	5.5	0.4718	2.4438	2.2189	5.5	0.7431	4.7776	5.868	5.5	1.315	2.1199			
2.0857	6.0	0.4813	2.7911	2.2888	6.0	0.7629	5.2367	5.986	6.0	1.386	2.5340			
2.1620	6.5	0.4877	3.1513	2.3479	6.5	0.7760	5.6844	6.094	6.5	1.454	2.9665			
2.2300	7.0	0.4920	3.5239	2.4025	7.0	0.7847	6.1262	6.195	7.0	1.519	3.4138			
2.2906	7.5	0.4948	3.9084	2.4489	7.5	0.7903	6.5663	6.287	7.5	1.578	3.8726			
2.3446	8.0	0.4967	4.3041	2.4844	8.0	0.7939	7.0075	6.372	8.0	1.633	4.3405			
2.3928	8.5	0.4979	4.7102	2.5146	8.5	0.7961	7.4514	6.45						

TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER  
UNSTABLE PLANT WITH ZERO - Continued

$\tau_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2$	$\tau_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2$	$\tau_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2$
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 2.0$					$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 5.0$					$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 2.0$				
0.8302	2.0	-0.2572	0.8541	1.3661	7.5	-0.7970	8.7526	0.6582	3.0	-0.4287	2.5411			
0.9567	2.5	-0.3274	1.2413	1.3706	8.0	-0.7982	9.2496	0.6723	3.5	-0.4559	3.0672			
1.0544	3.0	-0.3820	1.6552	1.3741	8.5	-0.7989	9.7462	0.6807	4.0	-0.4730	3.5846			
1.1293	3.5	-0.4217	2.0848	1.3768	9.0	-0.7993	10.2430	0.6856	4.5	-0.4835	4.0957			
1.1867	4.0	-0.4492	2.5249	1.3789	9.5	-0.7996	10.7401	0.6886	5.0	-0.4900	4.6027			
1.2310	4.5	-0.4675	2.9731	1.3805	10.0	-0.7997	11.2377	0.6904	5.5	-0.4939	5.1070			
1.2652	5.0	-0.4795	3.4285	1.3828	11.0	-0.7999	12.2339	0.6915	6.0	-0.4963	5.6096			
1.2918	5.5	-0.4872	3.8906	1.3842	12.0	-0.8000	13.2315	0.6921	6.5	-0.4977	6.1112			
1.3126	6.0	-0.4920	4.3589	1.3850	13.0	-0.8000	14.2299	0.6925	7.0	-0.4986	6.6122			
1.3288	6.5	-0.4951	4.8326	1.3855	14.0	-0.8000	15.2289	0.6928	7.5	-0.4992	7.1128			
1.3414	7.0	-0.4970	5.3111	1.3858	15.0	-0.8000	16.2283	0.6929	8.0	-0.4995	7.6131			
1.3513	7.5	-0.4981	5.7937	1.3860	16.0	-0.8000	17.2280	0.6930	8.5	-0.4997	8.1134			
1.3590	8.0	-0.4989	6.2797	1.3861	17.0	-0.8000	18.2278	0.6931	9.0	-0.4998	8.6135			
1.3650	8.5	-0.4993	6.7686	1.3862	18.0	-0.8000	19.2276							
1.3697	9.0	-0.4996	7.2598											
1.3734	9.5	-0.4997	7.7528											
1.3762	10.0	-0.4998	8.2473											
1.3802	11.0	-0.4999	9.2395											
1.3826	12.0	-0.5000	10.2348											
1.3840	13.0	-0.5000	11.2319											
1.3849	14.0	-0.5000	12.2301											
1.3855	15.0	-0.5000	13.2291											
1.3858	16.0	-0.5000	14.2284											
1.3860	17.0	-0.5000	15.2280											
1.3861	18.0	-0.5000	16.2278											
1.3862	19.0	-0.5000	17.2276											
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 3.0$					$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 0$					$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 3.0$				
0.5017	-1	-1.1365	-2	0.3748	-2	0.7501	-1	0.3121	-4	0.4953	-1	0.1	-0.2183	-2
0.1006	-2	-0.5362	-2	0.1497	-1	0.4994	-1	0.1501	0.2	0.9790	-1	0.2	-0.8564	-2
0.1510	-3	-0.1183	-1	0.3358	-1	0.7479	-1	0.2252	0.3	0.1448	0.3	0.3	-0.1885	-1
0.2012	-4	-0.2061	-1	0.5947	-1	0.9950	-1	0.3005	0.4	0.1900	0.4	0.4	-0.3273	-1
0.2510	-5	-0.3153	-1	0.9249	-1	1.2400	-1	0.3760	0.5	0.2333	0.5	0.5	-0.4984	-1
0.3004	-6	-0.4440	-1	1.324	-1	1.483	-1	0.4517	0.6	0.2744	0.6	0.6	-0.6977	-1
0.3490	-7	-0.5904	-1	1.791	-1	1.723	-1	0.5277	0.7	0.3132	0.7	0.7	-0.9213	-1
0.3968	-8	-0.7527	-1	2.321	-1	1.960	-1	0.6040	0.8	0.3495	0.8	0.8	-1.1165	-1
0.4437	-9	-0.9290	-1	2.913	-1	2.194	-1	0.6806	0.9	0.3833	0.9	0.9	-1.424	-1
0.4894	-10	-1.1117	-1	3.564	-1	2.423	-1	0.7577	1.0	0.4146	1.0	1.0	-1.694	-1
0.5361	-11	-1.2175	-1	4.275	-1	2.646	-1	0.8337	1.1	0.4437	1.1	1.1	-2.0691	-1
0.5822	-12	-1.3421	-1	5.049	-1	2.864	-1	0.9054	1.2	0.4691	1.2	1.2	-2.5130	-1
0.6281	-13	-1.4841	-1	5.887	-1	3.075	-1	0.9708	1.3	0.4915	1.3	1.3	-3.0708	-1
0.6739	-14	-1.6424	-1	6.792	-1	3.280	-1	1.0284	1.4	0.5108	1.4	1.4	-3.6676	-1
0.7196	-15	-1.8175	-1	7.764	-1	3.483	-1	1.0785	1.5	0.5270	1.5	1.5	-4.3055	-1
0.7654	-16	-2.0092	-1	8.807	-1	3.685	-1	1.1214	1.6	0.5408	1.6	1.6	-4.9890	-1
0.8112	-17	-2.2175	-1	9.919	-1	3.887	-1	1.1583	1.7	0.5523	1.7	1.7	-5.7203	-1
0.8570	-18	-2.4421	-1	11.100	-1	4.089	-1	1.1892	1.8	0.5617	1.8	1.8	-6.5034	-1
0.9028	-19	-2.6841	-1	12.350	-1	4.291	-1	1.2151	1.9	0.5689	1.9	1.9	-7.3401	-1
0.9486	-20	-2.9441	-1	13.670	-1	4.493	-1	1.2370	2.0	0.5740	2.0	2.0	-8.2342	-1
0.9944	-21	-3.2224	-1	15.060	-1	4.695	-1	1.2550	2.1	0.5778	2.1	2.1	-9.1817	-1
1.0402	-22	-3.5191	-1	16.520	-1	4.897	-1	1.2691	2.2	0.5803	2.2	2.2	-10.1870	-1
1.0860	-23	-3.8341	-1	18.050	-1	5.099	-1	1.2792	2.3	0.5817	2.3	2.3	-11.2542	-1
1.1318	-24	-4.1675	-1	19.650	-1	5.299	-1	1.2863	2.4	0.5823	2.4	2.4	-12.3870	-1
1.1776	-25	-4.5191	-1	21.320	-1	5.499	-1	1.2896	2.5	0.5817	2.5	2.5	-13.5800	-1
1.2234	-26	-4.8889	-1	23.060	-1	5.699	-1	1.2847	2.6	0.5798	2.6	2.6	-14.8370	-1
1.2692	-27	-5.2770	-1	24.870	-1	5.899	-1	1.2768	2.7	0.5766	2.7	2.7	-16.1620	-1
1.3150	-28	-5.6835	-1	26.750	-1	6.099	-1	1.2650	2.8	0.5723	2.8	2.8	-17.5590	-1
1.3608	-29	-6.1085	-1	28.700	-1	6.299	-1	1.2491	2.9	0.5670	2.9	2.9	-19.0320	-1
1.4066	-30	-6.5520	-1	30.720	-1	6.499	-1	1.2292	3.0	0.5608	3.0	3.0	-20.5860	-1
1.4524	-31	-7.0141	-1	32.810	-1	6.699	-1	1.2053	3.1	0.5537	3.1	3.1	-22.2250	-1
1.4982	-32	-7.4951	-1	34.960	-1	6.899	-1	1.1774	3.2	0.5458	3.2	3.2	-23.9520	-1
1.5440	-33	-8.0000	-1	37.180	-1	7.099	-1	1.1455	3.3	0.5370	3.3	3.3	-25.7710	-1
1.5898	-34	-8.5291	-1	39.470	-1	7.299	-1	1.1096	3.4	0.5273	3.4	3.4	-27.6860	-1
1.6356	-35	-9.0822	-1	41.830	-1	7.499	-1	1.0697	3.5	0.5168	3.5	3.5	-29.6920	-1
1.6814	-36	-9.6593	-1	44.260	-1	7.699	-1	1.0258	3.6	0.5055	3.6	3.6	-31.7940	-1
1.7272	-37	-10.2604	-1	46.760	-1	7.899	-1	0.9779	3.7	0.4935	3.7	3.7	-33.9970	-1
1.7730	-38	-10.8855	-1	49.330	-1	8.099	-1	0.9260	3.8	0.4808	3.8	3.8	-36.3050	-1
1.8188	-39	-11.5346	-1	51.960	-1	8.299	-1	0.8701	3.9	0.4675	3.9	3.9	-38.7140	-1
1.8646	-40	-12.2077	-1	54.650	-1	8.499	-1	0.8102	4.0	0.4537	4.0	4.0	-41.2290	-1
1.9104	-41	-12.9048	-1	57.400	-1	8.699	-1	0.7463	4.1	0.4394	4.1	4.1	-43.8560	-1
1.9562	-42	-13.6269	-1	60.210	-1	8.899	-1	0.6784	4.2	0.4247	4.2	4.2	-46.5900	-1
2.0020	-43	-14.3740	-1	63.080	-1	9.099	-1	0.6065	4.3	0.4097	4.3	4.3	-49.4360	-1
2.0478	-44	-15.1461	-1	66.010	-1	9.299	-1	0.5306	4.4	0.3947	4.4	4.4	-52.3980	-1
2.0936	-45	-15.9432	-1	69.000	-1	9.499	-1	0.4507	4.5	0.3794	4.5	4.5	-55.4800	-1
2.1394	-46	-16.7653	-1	72.050	-1	9.699	-1	0.3668	4.6	0.3637	4.6	4.6	-58.6860	-1
2.1852	-47	-17.6124	-1	75.160	-1	9.899	-1	0.2789	4.7	0.3477	4.7	4.7	-62.0120	-1
2.2310	-48	-18.4845	-1	78.330	-1	10.099	-1	0.1870	4.8	0.3314	4.8	4.8	-65.4540	-1
2.2768	-49	-19.3816	-1	81.560	-1	10.299	-1	0.0911	4.9	0.3147	4.9	4.9	-69.0160	-1
2.3226	-50	-20.3037	-1	84.850	-1	10.499	-1		5.0	0.2977	5.0	5.0	-72.6940	-1
2.3684	-51	-21.2508	-1	88.200	-1	10.699	-1		5.1	0.2804	5.1	5.1	-76.4920	-1
2.4142	-52	-22.2229	-1	91.610	-1	10.899	-1		5.2	0.2627	5.2	5.2	-80.4140	-1
2.4600	-53	-23.2200	-1	95.080	-1	11.099	-1		5.3	0.2447	5.3	5.3	-84.4640	-1
2.5058	-54	-24.2421	-1	98.610	-1	11.299	-1		5.4	0.2264	5.4	5.4	-88.6460	-1
2.5516	-55</													

TABLE III.- NORMALIZED OPTIMUM RESPONSE FOR A TYPE 1 THIRD-ORDER  
UNSTABLE PLANT WITH ZERO - Concluded

$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	b/B	$R_0/BK\tau_2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	b/B	$R_0/BK\tau_2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_m/\tau_2$	b/B	$R_0/BK\tau_2$
$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 0.5$					$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 3.0$					$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 0.5$				
0.4649 -1		0.1	0.9051 -2	0.2501 -2	0.2962		1.0	-0.2523	1.1647	0.1269		0.6	0.3027	0.1949
.8678 -1		.2	.3281 -1	.1002 -1	.3305		1.5	-.4014	2.0432	.1312		.7	.3642	.2556
.1219		.3	.6706 -1	.2262 -1	.3414		2.0	-.5030	2.8263	.1339		.8	.4219	.3212
.1527		.4	.1085	.4037 -1	.3448		2.5	-.5669	3.5109	.1357		.9	.4753	.3910
.1796		.5	.1547	.6334 -1	.3460		3.0	-.6060	4.1261	.1369		1.0	.5243	.4643
.2033		.6	.2038	.9156 -1	.3464		3.5	-.6299	4.6969	.1385		1.5	.7106	.8678
.2240		.7	.2541	.1250	.3465		4.0	-.6443	5.2400	.1386		2.0	.8244	1.3106
.2420		.8	.3046	.1637	$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 5.0$					$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 2.0$				
.2577		.9	.3544	.2074	.4839 -1		.1	-.4345 -2	.2494 -1	.4427 -1		.1	-.6625 -2	.2471 -1
.2712		1.0	.4029	.2561	.9319 -1		.2	-.1707 -1	.9897 -1	.7720 -1		.2	-.2493 -1	.9545 -1
.3151		1.5	.6116	.5640	.1339		.3	-.3751 -1	.2197	.1001		.3	-.5162 -1	.2031
.3341		2.0	.7973	.9531	.1702		.4	-.6471 -1	.3831	.1151		.4	-.8311 -1	.3361
.3418		2.5	.8810	1.3909	.2020		.5	-.9753 -1	.5836	.1245		.5	-.1163	.4838
.3448		3.0	.9092	1.8558	.2293		.6	-.1346	.8146	.1302		.6	-.1302	.6381
.3459		3.5	.9448	2.3358	.2523		.7	-.1747	1.0688	.1336		.7	-.1807	.7941
.3463		4.0	.9665	2.8241	.2714		.8	-.2164	1.3389	.1357		.8	-.2100	.9487
.3465		4.5	.9797	3.3172	.2870		.9	-.2585	1.6183	.1369		.9	-.2371	1.1003
$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 2.0$					.2896		1.0	-.3001	1.9010	.1376		1.0	-.2618	1.2484
.4805 -1		.1	-.3040 -2	.9977 -2	.3327		1.5	-.4806	3.2373	.1386		1.5	-.3553	1.9334
.9196 -1		.2	-.1177 -1	.3963 -1	.3425		2.0	-.6033	4.3316	$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 3.0$				
.1315		.3	-.2551 -1	.8811 -1	.3453		2.5	-.6802	5.2103	.4443 -1		.1	-.8635 -2	.3705 -1
.1664		.4	-.4345 -1	.1540	.3462		3.0	-.7272	5.9438	.7763 -1		.2	-.3272 -1	.1429
.1970		.5	-.6470 -1	.2355	.3464		3.5	-.7558	6.5863	.1007		.3	-.6815 -1	.3031
.2232		.6	-.8836 -1	.3304	.3465		4.0	-.7732	7.1730	.1157		.4	-.1101	.4990
.2454		.7	-.1136	.4363	$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 0$					.1251		.5	-.1546	.7136
.2641		.8	-.1395	.5509	.2372 -1	0.7379 -1	.1		.1558 -3	.1307		.6	-.1986	.9345
.2796		.9	-.1656	.6720	.4476 -1	.1454	.2		.1238 -2	.1340		.7	-.2407	1.1540
.2923		1.0	-.1912	.7977	.6309 -1	.2151	.3		.4128 -2	.1359		.8	-.2799	1.3677
.3280		1.5	-.3018	1.4475	.7873 -1	.2835	.4		.9625 -2	.1371		.9	-.3160	1.5738
.3402		2.0	-.3775	2.0745	.9182 -1	.3510	.5		.1842 -1	.1377		1.0	-.3490	1.7715
.3444		2.5	-.4252	2.6617	.1026	.4181	.6		.3105 -1	.1386		1.5	-.4737	2.6440
.3458		3.0	-.4545	3.2175	.1113	.4853	.7		.4794 -1	$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 5.0$				
.3463		3.5	-.4724	3.7522	.1182	.5529	.8		.6937 -1	.4455 -1		.1	-.1017 -1	.6174 -1
.3465		4.0	-.4833	4.2736	.1235	.6213	.9		.9547 -1	.7799 -1		.2	-.3876 -1	.2378
.3465		4.5	-.4898	4.7866	.1276	.6908	1.0		.1263	.1012		.3	-.8110 -1	.5031
$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 3.0$					.1368	1.0604	1.5		.3472	.1163		.4	-.1315	.8249
.4824 -1		.1	-.3815 -2	.1496 -1	.1384	1.4686	2.0		.6603	.1256		.5	-.1849	1.1736
.9263 -1		.2	-.1489 -1	.5941 -1	.1386	1.9080	2.5		1.0389	.1311		.6	-.2380	1.5276
.1328		.3	-.3251 -1	.1320	$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 0.5$					.1343		.7	-.2885	1.8740
.1685		.4	-.5577 -1	.2303	.4297 -1		.1	.1571 -1	.6208 -2	.1362		.8	-.3357	2.2060
.1997		.5	-.8359 -1	.3514	.7385 -1		.2	.5577 -1	.2442 -1	.1372		.9	-.3791	2.5209
.2265		.6	-.1148	.4916	.9548 -1		.3	.1107	.5367 -1	.1378		1.0	-.4187	2.8179
.2491		.7	-.1483	.6468	.1103		.4	.1733	.9271 -1	.1386		1.5	-.5684	4.0650
.2680		.8	-.1830	.8131	.1203		.5	.2384	.1402					
.2835		.9	-.2179	.9868										

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT WITH ZERO

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$
$\zeta = 0.10; \omega_n T_1 = 0$						$\zeta = 0.10; \omega_n T_1 = 4.0$					
0.5	1.5061	2.4715	2.9308		0.2050	0.5000 -1	0.1499		0.2	-0.6176 -4	0.1001 -2
1.0	2.4965	3.8584	4.7240		1.5036	.1003	.2993		.4	-.4829 -3	.8021 -2
1.5	3.1995	4.6451	5.8914		3.7642	.1511	.4479		.6	-.1595 -2	.2716 -1
2.0	3.7771	5.1630	6.7719		6.5393	.2030	.5956		.8	-.3705 -2	.6467 -1
2.5	4.3012	5.5559	7.5092		9.5765	.2563	.7421		1.0	-.7103 -2	.1271
3.0	4.8091	5.8930	8.1675		12.7548	.3083	1.1022		1.5	-.2302 -1	.4377
3.5	5.3273	6.2178	8.7810		16.0227	.3586	1.4529		2.0	-.5282 -1	1.0666
4.0	5.8847	6.5713	9.3733		19.3772	.7457	1.7945		2.5	-.1006	2.1543
4.5	6.5280	7.0118	9.9677		22.8769	.9699	2.1292		3.0	-.1704	3.8633
5.0	7.3584	7.6569	10.5972		26.6670						
5.5	8.6154	8.7735	11.3163		31.0297						
6.0	10.8398	10.9577	12.2357		36.2864						
6.5	13.0501	13.8887	14.0775		42.4894						
7.0	14.1977	15.0782	15.7614		49.4334						
7.5	15.1214	16.1535	17.0642		57.0636						
8.0	15.9189	16.9835	18.1293		65.3068						
8.5	16.6414	17.6578	19.0328		74.0104						
9.0	17.3310	18.2503	19.8386		83.0742						
9.5	18.0204	18.8143	20.5881		92.4397						
10.0	18.7402	19.3962	21.3118		102.0811						
10.5	19.5319	20.0508	22.0377		112.0403						
11.0	20.4650	20.8632	22.7966		122.3928						
11.5	21.6020	21.9130	23.6220		133.2165						
12.0	23.0171	23.3084	24.5827		144.6568						
12.5	24.5443	24.9449	25.8014		156.7788						
13.0	26.3433	26.4370	27.2056		169.5661						
13.5	26.9012	27.6595	28.5166		182.9750						
14.0	27.8381	28.6714	29.6669		196.9730						
14.5	28.7013	29.5481	30.6934		211.4666						
15.0	29.5228	30.3358	31.6259		226.4358						
$\zeta = 0.10; \omega_n T_1 = 0.5$						$\zeta = 0.10; \omega_n T_1 = 8.0$					
.4802 -1	.1475		.2	-0.1890 -2	.1253 -3	.5016 -1	.1500		.2	-.3059 -4	.2001 -2
.9299 -1	.2899		.4	-.1253 -1	.1008 -2	.1009	.2999		.4	-.2422 -3	.1604 -1
.1360	.4272		.6	-.3536 -1	.3443 -2	.1525	.4493		.6	-.8097 -3	.5430 -1
.1780	.5603		.8	-.7084 -1	.8299 -2	.2055	.5979		.8	-.1904 -2	.1293
.2196	.6901		1.0	-.1181	.1657 -1	.2602	.7455		1.0	-.3693 -2	.2539
.3252	1.0049		1.5	-.2810	.6016 -1	.4075	1.1083		1.5	-.1231 -1	.8727
.4383	1.3138		2.0	-.4979	.1564	.5759	1.4600		2.0	-.2897 -1	2.1176
.7018	1.9204	2.9686	3.0		.6328	.7741	1.7956		2.5	-.5637 -1	4.2458
.9802	2.4272	3.6970	4.0		1.5125	1.0122	2.1242		3.0	-.9702 -1	7.5252
$\zeta = 0.10; \omega_n T_1 = 1.0$						$\zeta = 0.10; \omega_n T_1 = 16.0$					
.4911 -1	.1489		.2	-.4078 -3	.2503 -3	.5023 -1	.1501		.2	-.1540 -4	.4003 -2
.9684 -1	.2954		.4	-.2964 -2	.2008 -2	.1012	.3002		.4	-.1227 -3	.3208 -1
.1437	.4392		.6	-.9123 -2	.6812 -2	.1532	.4499		.6	-.4128 -3	.1806
.1903	.5804		.8	-.1979 -1	.1627 -1	.2068	.5990		.8	-.9764 -3	.2585
.2370	.7192		1.0	-.3550 -1	.3211 -1	.2623	.7471		1.0	-.1905 -2	.5075
.3568	1.0575		1.5	-.9853 -1	.1124	.4125	1.1109		1.5	-.6442 -2	1.7427
.4862	1.3877		2.0	-.1971	.2804	.5852	1.4624		2.0	-.1536 -1	4.2186
.8023	2.0467		3.0	-.5111	1.0897	.7894	1.7979		2.5	-.3018 -1	8.4184
1.2517	2.7405	3.9887	4.0		3.0332	1.0344	2.1160		3.0	-.5229 -1	14.8052
$\zeta = 0.10; \omega_n T_1 = 2.0$						$\zeta = 0.25; \omega_n T_1 = 0$					
.4970 -1	.1496		.2	-.1369 -3	.5004 -3	.5	1.4949	2.3961	2.8024		.1702
.9906 -1	.2980		.4	-.1044 -2	.4012 -2	1.0	2.5448	3.8009	4.5122		1.2094
.1485	.4451		.6	-.3367 -2	.1359 -1	1.5	3.3615	4.6917	5.6602		3.0567
.1984	.5907		.8	-.7639 -2	.3238 -1	2.0	4.0807	5.3559	6.5502		5.4192
.2491	.7348		1.0	-.1431 -1	.6369 -1	2.5	4.7676	5.9242	7.3131		8.1249
.3820	1.0881		1.5	-.4390 -1	.2204	3.0	5.4624	6.4699	8.0149		11.0983
.5291	1.4332		2.0	-.9584 -1	.5409	3.5	6.2011	7.0496	8.6971		14.3261
.8981	2.1124		3.0	-.2857	2.0128	4.0	7.0135	7.7072	9.3872		17.8490
1.4432	2.8184		4.0	-.6248	5.3583	4.5	7.9460	8.5061	10.1201		21.7439
$\zeta = 0.25; \omega_n T_1 = 0.5$						5.0	9.0425	9.5100	10.9349		26.1118
.4836 -1	.1479		.2	-.1644 -2	.1254 -3	5.5	10.3104	10.7539	11.8871		31.0588
.9425 -1	.2915		.4	-.1090 -1	.1008 -2	6.0	11.6378	12.1510	13.0263		36.6344
.1387	.4310		.6	-.3076 -1	.3440 -2	6.5	12.8284	13.4541	14.2514		42.8564
.1824	.5670		.8	-.6159 -1	.8285 -2	7.0	13.9272	14.6581	15.4617		49.6402
.2261	.7005		1.0	-.1026	.1652 -1	7.5	14.9338	15.7335	16.5994		56.9950
.3385	1.0277		1.5	-.2430	.5964 -1	8.0	15.8895	16.7188	17.6987		64.8703
						8.5	16.8127	17.6405	18.6553		73.2419
						9.0	17.7023	18.4919	19.5792		82.0943
						9.5	18.6491	19.3934	20.4885		91.3236
						10.0	19.6565	20.3853	21.4574		101.0258

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT  
WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$
$\zeta = 0.25; \omega_n T_1 = 0.5$						$\zeta = 0.50; \omega_n T_1 = 0$					
0.4602	1.3535		2.0	-0.4269	0.1538	0.5	1.4860	2.3159	2.6598		0.1319
.5972	1.6853		2.5	-.6477	.3301	1.0	2.6095	3.7520	4.2847		.9122
.9124	2.3380	3.4256	3.5		1.0240	1.5	3.5710	4.8085	5.4752		2.3589
1.2745	2.9615	4.1870	4.5		2.2400	2.0	4.4574	5.6844	6.4538		4.3825
1.7534	3.6757	4.9223	5.5		4.3324	2.5	5.3274	6.4958	7.3356		6.8496
2.3530	4.4955	5.6425	6.5		7.4602	3.0	6.2052	7.2951	8.1798		9.7408
3.0456	5.4343	6.3887	7.5		11.5655	3.5	7.1083	8.1201	9.0234		13.0538
$\zeta = 0.25; \omega_n T_1 = 1.0$						4.0	8.1246	9.0985	9.9477		16.7050
.4947 -1	.1493		.2	-.3399 -3	.2503 -3	4.5	9.1587	10.1119	10.9063		20.8961
.9521 -1	.2970		.4	-.2471 -2	.2008 -2	5.0	10.1779	11.1221	11.8889		25.5598
.1467	.4429		.6	-.7603 -2	.6809 -2	5.5	11.1779	12.1221	12.8889		30.8107
.1954	.5872		.8	-.1649 -1	.1625 -1	$\zeta = 0.50; \omega_n T_1 = 0.5$					
.2447	.7299		1.0	-.2957 -1	.3204 -1	.4893 -1	.1486		.2	-0.1233 -2	.1253 -3
.3732	1.0823		1.5	-.8190 -1	.1118	.9634 -1	.2943		.4	-.8171 -2	.1007 -2
.5141	1.4326		2.0	-.1631	.2771	.1430	.4373		.6	-.2305 -1	.3432 -2
.8598	2.1526		3.0	-.4144	1.0519	.1897	.5782		.8	-.4608 -1	.8244 -2
1.3414	2.9452		4.0	-.7925	2.8370	.2368	.7176		1.0	-.7658 -1	.1638 -1
$\zeta = 0.25; \omega_n T_1 = 2.0$						.3594	1.0646		1.5	-.1792	.5836 -1
.5006 -1	.1499		.2	-.1190 -3	.5005 -3	.4931	1.4158		2.0	-.3090	.1477
.1005	.2996		.4	-.9078 -2	.4011 -2	.6424	1.7782		2.5	-.4568	.3094
.1517	.4487		.6	-.2927 -2	.1358 -1	1.0016	2.5552		3.5	-.7860	.9725
.2039	.5973		.8	-.6639 -2	.3235 -1	1.4287	3.3730	4.4443	4.5		2.2092
.2577	.7452		1.0	-.1243 -1	.6358 -1	1.9007	4.2134	5.3127	5.5		4.0405
.4010	1.1130		1.5	-.3806 -1	.2193	2.4417	5.1499	6.2082	6.5		6.6428
.5626	1.4798		2.0	-.8281 -1	.5357	3.0215	6.1639	7.1424	7.5		9.9752
.9723	2.2292		3.0	-.2431	1.9547	$\zeta = 0.5; \omega_n T_1 = 1.0$					
1.5593	3.0458		4.0	-.5136	5.0079	.5006 -1	.1499		.2	-.2255 -3	.2502 -3
2.3682	3.9920		5.0	-.8762	10.0944	.1005	.2997		.4	-.1646 -2	.2006 -2
$\zeta = 0.25; \omega_n T_1 = 4.0$						.1516	.4491		.6	-.5061 -2	.6793 -2
.5037 -1	.1503		.2	-.5720 -3	.1001 -2	.2038	.5983		.8	-.1096 -1	.1618 -1
.1017	.3008		.4	-.4470 -3	.8019 -2	.2573	.7475		1.0	-.1962 -1	.3182 -1
.1544	.4515		.6	-.1476 -2	.2714 -1	.3992	1.1222		1.5	-.5389 -1	.1098
.2089	.6020		.8	-.3428 -2	.6460 -1	.5561	1.5032		2.0	-.1058	.2678
.2654	.7523		1.0	-.6567 -2	.1268	.9323	2.3044		3.0	-.2559	.9645
.4190	1.1265		1.5	-.2123 -1	.4356	1.4110	3.1853		4.0	-.4514	2.4123
.5961	1.4990		2.0	-.4855 -1	1.0568	1.9889	4.1582		5.0	-.6612	4.8025
.8058	1.8719		2.5	-.9193 -1	2.1200	2.6256	5.2043		6.0	-.8426	8.0590
1.3646	2.6385		3.5	-.2380	6.1157	3.2678	6.2877		7.0	-.9604	12.0388
2.1689	3.4909		4.5	-.4640	13.1341	4.4440	8.4505		9.0	-.9871	21.0322
$\zeta = 0.25; \omega_n T_1 = 8.0$						5.4623	10.5277		11.0	-.8691	30.9252
.5053 -1	.1504		.2	-.2944 -4	.2002 -2	6.4124	12.5255		13.0	-.7717	42.0335
.1024	.3014		.4	-.2329 -3	.1604 -1	7.3684	14.4982		15.0	-.7406	55.2318
.1559	.4528		.6	-.7785 -3	.5426 -1	8.3518	16.4769		17.0	-.7498	70.7319
.2115	.6042		.8	-.1830 -2	.1291	9.3546	18.4705		19.0	-.7680	88.5147
.2597	.7555		1.0	-.3546 -2	.2534	10.3630	20.4734		21.0	-.7792	108.4113
.4292	1.1321		1.5	-.1179 -1	.8684	$\zeta = 0.5; \omega_n T_1 = 2.0$					
.6158	1.5052		2.0	-.2765 -1	2.0982	.5057 -1	.1506		.2	-.8924 -4	.5003 -3
.8392	1.8752		2.5	-.5348 -1	4.1800	.1029	.3022		.4	-.6803 -3	.4006 -2
1.4386	2.6195		3.5	-.1423	11.7570	.1569	.4547		.6	-.2191 -2	.1355 -1
$\zeta = 0.25; \omega_n T_1 = 16.0$						.2131	.6081		.8	-.4961 -2	.3220 -1
.5061 -1	.1505		.2	-.1511 -4	.4003 -2	.2717	.7625		1.0	-.9268 -2	.6312 -1
.1027	.3017		.4	-.1203 -3	.3207 -1	.4312	1.1530		1.5	-.2814 -1	.2156
.1566	.4534		.6	-.4047 -3	.1085	.6133	1.5529		2.0	-.6042 -1	.5186
.2129	.6052		.8	-.9567 -3	.2581	.8232	1.9658		2.5	-.1074	1.0286
.2719	.7569		1.0	-.1865 -2	.5064	1.3429	2.8475		3.5	-.2454	2.8793
.4347	1.1344		1.5	-.6290 -2	1.7339	1.9958	3.8137		4.5	-.4272	5.0272
.6265	1.5070		2.0	-.1493 -1	4.1791	2.7330	4.8672		5.5	-.6158	10.3320
.8572	1.8738		2.5	-.2918 -1	8.2885	4.1811	7.0561		7.5	-.8750	20.8069
1.1373	2.2359		3.0	-.5017 -1	14.4569	5.3879	9.2198		9.5	-.9181	31.9569
						6.3529	11.2329		11.5	-.8101	42.9159
						7.2309	13.2791		13.5	-.7018	54.6799
						8.1589	15.2434		15.5	-.6655	68.9510
						9.1455	17.2230		17.5	-.6735	86.1143
						10.1617	19.2198		19.5	-.6919	105.7668

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT  
 WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$
$\zeta = 0.5; \omega_n T_1 = 4.0$						$\zeta = 0.75; \omega_n T_1 = 1.0$					
0.5099 -1	0.1509		0.2	-0.4956 -4	0.1001 -2	0.5066 -1	0.1506		0.2	-0.1133 -3	0.2501 -3
.1042	.3034		.4	-.3870 -3	.8009 -2	.1027	.3023		.4	-.8220 -3	.2002 -2
.1599	.4573		.6	-.1276 -2	.2707 -1	.1565	.4552		.6	-.2522 -2	.6765 -2
.2185	.6126		.8	-.2958 -2	.6429 -1	.2120	.6093		.8	-.5448 -2	.1606 -1
.2803	.7690		1.0	-.5653 -2	.1259	.2695	.7646		1.0	-.9714 -2	.3144 -1
.4519	1.1657		1.5	-.1811 -1	.4281	.4228	1.1596		1.5	-.2633 -1	.1068
.6532	1.5715		2.0	-.4085 -1	1.0233	.5915	1.5661		2.0	-.5067 -1	.2549
.8908	1.9890		2.5	-.7590 -1	2.0110	.7768	1.9863		2.5	-.8110 -1	.5007
1.4959	2.8738		3.5	-.1860	5.4771	1.1982	2.8720		3.5	-.1525	1.3738
2.2707	3.8416		4.5	-.3395	11.0210	1.6785	3.8156		4.5	-.2258	2.8581
4.0268	5.9609		6.5	-.6580	25.8268	2.1976	4.8030		5.5	-.2891	4.9938
5.6545	8.1275		8.5	-.8885	42.3835	2.7339	5.8158		6.5	-.3361	7.7454
$\zeta = 0.5; \omega_n T_1 = 8.0$						$\zeta = 0.75; \omega_n T_1 = 2.0$					
.5115 -1	.1510		.2	-.2750 -4	.2001 -2	.5128 -1	.1512		.2	-.5949 -4	.5001 -3
.1048	.3040		.4	-.2174 -3	.1602 -1	.1053	.3048		.4	-.4527 -3	.3999 -2
.1614	.4585		.6	-.7256 -3	.5412 -1	.1621	.4607		.6	-.1455 -2	.1349 -1
.2214	.6146		.8	-.1702 -2	.1285	.2220	.6188		.8	-.3284 -2	.3196 -1
.2851	.7719		1.0	-.3290 -2	.2514	.2893	.7791		1.0	-.6111 -2	.6236 -1
.4638	1.1705		1.5	-.1083 -1	.8532	.4587	1.1904		1.5	-.1830 -1	.4937
.6768	1.5766		2.0	-.2505 -1	2.0313	.6599	1.6174		2.0	-.3849 -1	.1617
.9320	1.9918		2.5	-.4755 -1	3.9678	1.1243	2.5223		3.0	-.1020	1.6177
1.5923	2.8614		3.5	-.1202	10.5876	1.6790	3.4920		4.0	-.1870	3.6186
3.4139	4.8213		5.5	-.3356	32.7151	2.7857	4.5088		5.0	-.2770	6.4820
5.3321	6.9395		7.5	-.5356	58.0204	2.9062	5.5494		6.0	-.3568	10.0589
7.1241	9.0394		9.5	-.7087	85.0474	4.0957	7.6376		8.0	-.4580	18.7198
8.8832	11.1305		11.5	-.8757	115.4568	5.1814	9.6930		10.0	-.4883	28.8283
$\zeta = 0.5; \omega_n T_1 = 16.0$						$\zeta = 0.75; \omega_n T_1 = 4.0$					
.5123 -1	.1511		.2	-.1462 -4	.4002 -2	.5161 -1	.1515		.2	-.4191 -4	.9999 -3
.1052	.3042		.4	-.1163 -3	.3203 -1	.1066	.3059		.4	-.3269 -3	.7993 -2
.1622	.4591		.6	-.3906 -3	.1082	.1653	.4631		.6	-.1075 -2	.2695 -1
.2229	.6155		.8	-.9215 -3	.2569	.2279	.6229		.8	-.2484 -2	.6378 -1
.2875	.7732		1.0	-.1792 -2	.5025	.2947	.7853		1.0	-.4472 -2	.1243
.4701	1.1723		1.5	-.5982 -2	1.7031	.4821	1.2023		1.5	-.1492 -1	.4163
.6896	1.5776		2.0	-.1400 -1	4.0453	.7013	1.6354		2.0	-.3296 -1	.9737
.9544	1.9896		2.5	-.2689 -1	7.8704	1.2418	2.5519		3.0	-.9496 -1	3.1289
1.2716	2.4106		3.0	-.4512 -1	13.4113	1.9039	3.5313		4.0	-.1863	6.8073
3.0381	4.2541		5.0	-.1605	50.2963	2.6406	4.5560		5.0	-.2923	11.8074
5.0827	6.3265		7.0	-.2820	93.4738	4.1436	6.6586		7.0	-.4925	24.3369
7.0018	8.3965		9.0	-.3882	136.2389	5.5287	8.7604		9.0	-.6341	38.6541
$\zeta = 0.75; \omega_n T_1 = 0$						$\zeta = 0.75; \omega_n T_1 = 8.0$					
.5	1.4863	2.2708	2.5690		.1024	.5177 -1	.1517		.2	-.2556 -4	.2000 -2
1.0	2.6800	3.7856	4.2111		.7183	.1073	.3065		.4	-.2017 -3	.1598 -1
1.5	3.6975	4.8829	5.3709		2.0022	.1669	.4642		.6	-.6717 -3	.5387 -1
2.0	4.6743	5.8647	6.3807		3.8115	.2310	.6247		.8	-.1570 -2	.1274
2.5	5.6071	6.8049	7.3957		6.2306	.2999	.7878		1.0	-.3021 -2	.2483
3.0	6.6048	7.8037	8.3973		8.9842	.4955	1.2063		1.5	-.9798 -2	.8293
3.5	7.6033	8.8026	9.3983		12.2387	.7285	1.6397		2.0	-.2218 -1	1.9328
4.0	8.6001	9.7999	10.4000		16.0025	1.3173	2.5509		3.0	-.6662 -1	6.1338
4.5	9.6000	10.8000	11.4000		20.2513	2.0575	3.5178		4.0	-.1349	13.0935
5.0	10.6000	11.8000	12.4000		25.0000	3.7730	5.5592		6.0	-.3034	32.6550
5.5	11.6000	12.8000	13.4000		30.2500	5.5142	7.6431		8.0	-.4678	55.8863
$\zeta = 0.75; \omega_n T_1 = 0.5$						$\zeta = 0.75; \omega_n T_1 = 0.5$					
.4949 -1	.1493		.2	-.8219 -3	.1251 -3	.4949 -1	.1493		.2	-.8219 -3	.1251 -3
.9842 -1	.2971		.4	-.5442 -2	.1005 -2	.9842 -1	.2971		.4	-.5442 -2	.1005 -2
.1473	.4435		.6	-.1533 -1	.3417 -2	.1473	.4435		.6	-.1533 -1	.3417 -2
.1967	.5891		.8	-.3055 -1	.8179 -2	.1967	.5891		.8	-.3055 -1	.8179 -2
.2470	.7343		1.0	-.5058 -1	.1617 -1	.2470	.7343		1.0	-.5058 -1	.1617 -1
.3783	1.0992		1.5	-.1165	.5665 -1	.3783	1.0992		1.5	-.1165	.5665 -1
.5208	1.4717		2.0	-.1964	.1402	.5208	1.4717		2.0	-.1964	.1402
.8479	2.2559		3.0	-.3682	.5144	.8479	2.2559		3.0	-.3682	.5144
1.2366	3.1041		4.0	-.5302	1.2990	1.2366	3.1041		4.0	-.5302	1.2990
1.6814	4.0148		5.0	-.6664	2.6244	1.6814	4.0148		5.0	-.6664	2.6244
2.1676	4.9753		6.0	-.7695	4.5555	2.1676	4.9753		6.0	-.7695	4.5555
2.6777	5.9680		7.0	-.8388	7.0923	2.6777	5.9680		7.0	-.8388	7.0923
3.1972	6.9775		8.0	-.8788	10.1983	3.1972	6.9775		8.0	-.8788	10.1983
4.2337	9.0083		10.0	-.9018	17.9589	4.2337	9.0083		10.0	-.9018	17.9589
5.2528	11.0295		12.0	-.8931	27.6400	5.2528	11.0295		12.0	-.8931	27.6400
6.2586	13.0382		14.0	-.8816	39.2163	6.2586	13.0382		14.0	-.8816	39.2163
7.2587	15.0400		16.0	-.8751	52.7308	7.2587	15.0400		16.0	-.8751	52.7308
8.2576	17.0394		18.0	-.8728	68.2257	8.2576	17.0394		18.0	-.8728	68.2257
9.2566	19.0385		20.0	-.8725	85.7226	9.2566	19.0385		20.0	-.8725	85.7226
10.2562	21.0380		22.0	-.8728	105.2271	10.2562	21.0380		22.0	-.8728	105.2271

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT  
WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$
$\zeta = 0.75; \omega_n T_1 = 16.0$						$\zeta = 1.02; \omega_n T_1 = 1.0$					
0.5185 <sup>-1</sup>	0.1517		0.2	-0.1413 <sup>-4</sup>	0.3999 <sup>-2</sup>	4.2954	9.3083		10.0	0.2577 <sup>-1</sup>	17.9106
.1077	.3068		.4	-.1122 <sup>-3</sup>	.3196 <sup>-1</sup>	4.7913	10.3045		11.0	.2634 <sup>-1</sup>	22.4094
.1678	.4648		.6	-.3759 <sup>-3</sup>	.1077	5.2886	11.3019		12.0	.2673 <sup>-1</sup>	27.4173
.2326	.6256		.8	-.8837 <sup>-3</sup>	.2548	5.7868	12.3003		13.0	.2700 <sup>-1</sup>	32.9322
.3026	.7890		1.0	-.1700 <sup>-2</sup>	.4961	6.2856	13.2992		14.0	.2718 <sup>-1</sup>	38.9521
.5027	1.2077		1.5	-.5623 <sup>-2</sup>	1.6552	6.7848	14.2985		15.0	.2730 <sup>-1</sup>	45.4759
.7433	1.6402		2.0	-.1289 <sup>-1</sup>	3.8491	7.2843	15.2980		16.0	.2738 <sup>-1</sup>	52.5024
1.0285	2.0857		2.5	-.2411 <sup>-1</sup>	7.3027	7.7840	16.2977		17.0	.2744 <sup>-1</sup>	60.0308
1.3595	2.5439		3.0	-.3945 <sup>-1</sup>	12.1162	8.2838	17.2975		18.0	.2747 <sup>-1</sup>	68.0607
3.0397	4.4883		5.0	-.1314	42.5620	8.7836	18.2974		19.0	.2750 <sup>-1</sup>	76.5916
4.9321	6.5275		7.0	-.2381	81.3945	9.2835	19.2973		20.0	.2751 <sup>-1</sup>	85.6232
6.8108	8.5801		9.0	-.3413	123.2668	10.2834	20.2972		22.0	.2753 <sup>-1</sup>	105.1879
$\zeta = 1.01; \omega_n T_1 = 0$						11.2834	21.2971		24.0	.2754 <sup>-1</sup>	126.7535
.50	1.4669	2.1794	2.4250		.9230 <sup>-1</sup>	12.2833	22.2971		26.0	.2754 <sup>-1</sup>	150.3197
1.0	2.7049	3.7414	4.0729		.6130	13.2833	23.2971		28.0	.2754 <sup>-1</sup>	175.8861
1.5	3.8405	5.0533	5.4256		1.6816	14.2833	24.2971		30.0	.2755 <sup>-1</sup>	203.4526
2.0	4.9245	6.2440	6.6391		3.3028	$\zeta = 1.02; \omega_n T_1 = 2.0$					
2.5	5.9769	7.3633	7.7727		5.4634	.5194 <sup>-1</sup>	.1519		.2	-.2734 <sup>-4</sup>	.4996 <sup>-3</sup>
3.0	7.0020	8.4186	8.8332		8.1672	.1078	.3076		.4	-.2077 <sup>-3</sup>	.3987 <sup>-2</sup>
3.5	8.0289	9.4802	9.9025		11.3698	.1676	.4669		.6	-.6650 <sup>-3</sup>	.1340 <sup>-1</sup>
4.0	9.0365	10.4962	10.9195		15.1056	.2314	.6299		.8	-.1494 <sup>-2</sup>	.3158 <sup>-1</sup>
4.5	10.0448	11.5146	11.9399		19.3418	.2991	.7963		1.0	-.2764 <sup>-2</sup>	.6123 <sup>-1</sup>
5.0	11.0507	12.5290	12.9564		24.0757	.4849	1.2267		1.5	-.8107 <sup>-2</sup>	.2016
5.5	12.0585	13.5461	13.9754		29.3154	.6925	1.6759		2.0	-.1662 <sup>-1</sup>	.4620
6.0	13.0587	14.5463	14.9753		35.0660	1.1605	2.6191		3.0	-.4142 <sup>-1</sup>	1.4270
6.5	14.0588	15.5463	15.9752		41.3152	1.6742	3.6024		4.0	-.7176 <sup>-1</sup>	3.0356
7.0	15.0587	16.5464	16.9753		48.0634	2.2087	4.6065		5.0	-.1022	5.2722
7.5	16.0589	17.5469	17.9760		55.3118	2.7481	5.6189		6.0	-.1292	8.0870
8.0	17.0658	18.5632	18.9949		63.0523	3.2842	6.6329		7.0	-.1513	11.4330
$\zeta = 1.02; \omega_n T_1 = 0.5$						3.8140	7.6455		8.0	-.1685	15.2770
.5011 <sup>-1</sup>	.1500		.2	-.3775 <sup>-3</sup>	.1252 <sup>-3</sup>	4.3370	8.6558		9.0	-.1813	19.6004
.1006	.3000		.4	-.2499 <sup>-2</sup>	.1003 <sup>-2</sup>	4.8541	9.6636		10.0	-.1905	24.3953
.1519	.4501		.6	-.7016 <sup>-2</sup>	.3396 <sup>-2</sup>	5.3665	10.6694		11.0	-.1971	29.6608
.2040	.6006		.8	-.1393 <sup>-1</sup>	.8085 <sup>-2</sup>	5.8752	11.6736		12.0	-.2016	35.3994
.2573	.7516		1.0	-.2293 <sup>-1</sup>	.1588 <sup>-1</sup>	6.3812	12.6765		13.0	-.2047	41.6154
.3965	1.1335		1.5	-.5179 <sup>-1</sup>	.5446 <sup>-1</sup>	6.8854	13.6785		14.0	-.2069	48.3132
.5452	1.5239		2.0	-.8497 <sup>-1</sup>	.1313	7.3882	14.6799		15.0	-.2083	55.4968
.7874	2.3360		3.0	-.1495	.4558	7.8902	15.6809		16.0	-.2093	63.1697
1.2414	3.1910		4.0	-.2020	1.0954	8.3915	16.6815		17.0	-.2100	71.3345
1.6439	4.0836		5.0	-.2410	2.1365	8.8923	17.6819		18.0	-.2104	79.9934
2.0738	5.0066		6.0	-.2687	3.6391	9.3929	18.6822		19.0	-.2107	89.1480
2.5246	5.9526		7.0	-.2881	5.6399	9.8933	19.6824		20.0	-.2109	98.7996
2.9905	6.9152		8.0	-.3015	8.1575	10.3938	20.6826		22.0	-.2111	119.5967
3.4672	7.8896		9.0	-.3106	11.1992	11.8940	23.6827		24.0	-.2112	142.3893
3.9514	8.8722		10.0	-.3168	14.7658	12.8940	25.6828		26.0	-.2113	167.1796
4.4407	9.8605		11.0	-.3210	18.8549	13.8941	27.6828		28.0	-.2113	193.9688
4.9336	10.8526		12.0	-.3239	23.4631	14.8941	29.6828		30.0	-.2113	222.7575
5.4288	11.8473		13.0	-.3258	28.5865	$\zeta = 1.02; \omega_n T_1 = 4.0$					
5.9256	12.8438		14.0	-.3271	34.2219	.5227 <sup>-1</sup>	.1522		.2	-.3364 <sup>-4</sup>	.9988 <sup>-3</sup>
6.4234	13.8414		15.0	-.3279	40.3663	.1092	.3087		.4	-.2619 <sup>-3</sup>	.7969 <sup>-2</sup>
6.9220	14.8398		16.0	-.3285	47.0176	.1710	.4693		.6	-.8584 <sup>-3</sup>	.2676 <sup>-1</sup>
7.4210	15.8388		17.0	-.3289	54.1738	.2377	.6337		.8	-.1973 <sup>-2</sup>	.6301 <sup>-1</sup>
7.9204	16.8381		18.0	-.3292	61.8337	.3094	.8020		1.0	-.3731 <sup>-2</sup>	.1220
8.4200	17.8376		19.0	-.3293	69.9963	.5109	1.2378		1.5	-.1153 <sup>-1</sup>	.3999
8.9197	18.8373		20.0	-.3295	78.6608	.7428	1.6932		2.0	-.2481 <sup>-1</sup>	.9106
9.4194	20.8370		22.0	-.3296	97.4934	1.2855	2.6507		3.0	-.6738 <sup>-1</sup>	2.7605
10.9192	22.8368		24.0	-.3296	118.3289	1.9016	3.6502		4.0	-.1257	5.7302
11.9192	24.8368		26.0	-.3297	141.1659	2.5534	4.6713		5.0	-.1910	9.6788
12.9191	26.8367		28.0	-.3297	166.0035	3.2130	5.7008		6.0	-.2561	14.4199
13.9191	28.8367		30.0	-.3297	192.8414	3.8636	6.7314		7.0	-.3161	19.7931
$\zeta = 1.02; \omega_n T_1 = 1.0$						4.4964	7.7595		8.0	-.3685	25.6781
.5130 <sup>-1</sup>	.1513		.2	.9033 <sup>-5</sup>	.2500 <sup>-3</sup>	5.1077	8.7835		9.0	-.4121	31.9904
.1052	.3052		.4	.6559 <sup>-4</sup>	.1996 <sup>-2</sup>	5.6971	9.8031		10.0	-.4470	38.6765
.1616	.4617		.6	.2007 <sup>-3</sup>	.6721 <sup>-2</sup>	6.2664	10.8184		11.0	-.4740	45.7100
.2205	.6207		.8	.4315 <sup>-3</sup>	.1588 <sup>-1</sup>	6.8183	11.8299		12.0	-.4942	53.0865
.2818	.7822		1.0	.7651 <sup>-3</sup>	.3088 <sup>-1</sup>	7.3563	12.8384		13.0	-.5089	60.8167
.4453	1.1963		1.5	.2035 <sup>-2</sup>	.1028	7.8833	13.8446		14.0	-.5194	68.9205
.6221	1.6240		2.0	.3819 <sup>-2</sup>	.2390	8.4023	14.8489		15.0	-.5267	77.4207
1.0100	2.5141		3.0	.8216 <sup>-2</sup>	.7679	8.9154	15.8519		16.0	-.5318	86.3393
1.4335	3.4399		4.0	.1271 <sup>-1</sup>	1.7143	9.4244	16.8539		17.0	-.5353	95.6959
1.8823	4.3906		5.0	.1662 <sup>-1</sup>	3.1348	9.9305	17.8553		18.0	-.5376	105.5061
2.3484	5.3582		6.0	.1972 <sup>-1</sup>	5.0578	10.4346	18.8563		19.0	-.5392	115.7824
2.8260	6.3371		7.0	.2205 <sup>-1</sup>	7.4495	10.9374	19.8569		20.0	-.5403	126.5343
3.3114	7.3232		8.0	.2374 <sup>-1</sup>	10.4510	11.4405	20.8576		22.0	-.5415	149.4905
3.8017	8.3142		9.0	.2494 <sup>-1</sup>	13.9235	12.9419	23.8579		24.0	-.5420	174.4106
						13.9425	25.8581		26.0	-.5422	201.3130
						14.9428	27.8582		28.0	-.5423	230.2069
						15.9429	29.8582		30.0	-.5424	261.0966

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT  
 WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$
$\zeta = 1.02; \omega_n T_1 = 8.0$						$\zeta = 1.5; \omega_n T_1 = 0$					
0.5244 <sup>-1</sup>	0.1523		0.2	-0.2345 <sup>-4</sup>	0.1998 <sup>-2</sup>	7.5	16.7713	18.8055	19.6684		53.1826
.1099	.3092		.4	-.1847 <sup>-3</sup>	.1593 <sup>-1</sup>	8.0	17.7839	19.8306	20.0936		60.8881
.1728	.4703		.6	-.6127 <sup>-3</sup>	.5349 <sup>-1</sup>	8.5	18.7953	20.8536	21.1166		69.0974
.2411	.6354		.8	-.1425 <sup>-2</sup>	.1259	9.0	19.8032	21.8695	22.1325		77.8152
.3151	.8042		1.0	-.2724 <sup>-2</sup>	.2436	9.5	20.8115	22.8861	23.1492		87.0380
.5259	1.2413		1.5	-.8648 <sup>-2</sup>	.7964	10.0	21.8176	23.8982	24.1613		96.7672
.7733	1.6970		2.0	-.1907 <sup>-1</sup>	1.8074	$\zeta = 1.5; \omega_n T_1 = 0.5$					
1.3684	2.6519		3.0	-.5414 <sup>-1</sup>	5.4243	.5119 <sup>-1</sup>	.1513		.2	0.4098 <sup>-3</sup>	.1252 <sup>-3</sup>
2.0663	3.6460		4.0	-.1051	11.1110	.1045	.3052		.4	.2700 <sup>-2</sup>	.9964 <sup>-3</sup>
2.8248	4.6615		5.0	-.1658	18.4949	.1596	.4615		.6	.7527 <sup>-2</sup>	.3342 <sup>-2</sup>
3.6105	5.6870		6.0	-.2309	27.1654	.2161	.6198		.8	.1479 <sup>-1</sup>	.7865 <sup>-2</sup>
4.4024	6.7161		7.0	-.2966	36.8144	.2739	.7799		1.0	.2403 <sup>-1</sup>	.1523 <sup>-1</sup>
5.1880	7.7457		8.0	-.3606	47.2332	.4227	1.1857		1.5	.5217 <sup>-1</sup>	.5014 <sup>-1</sup>
5.9597	8.7743		9.0	-.4214	58.2750	.5765	1.5970		2.0	.8180 <sup>-1</sup>	.1158
6.7128	9.8010		10.0	-.4780	69.8233	.8978	2.4306		3.0	.1316	.3718
7.4436	10.8254		11.0	-.5296	81.7747	1.2369	3.2782		4.0	.1651	.8452
8.1494	11.8471		12.0	-.5756	94.0330	1.5942	4.1408		5.0	.1863	1.5923
8.8282	12.8660		13.0	-.6156	106.5100	1.9691	5.0190		6.0	.1999	2.6639
9.4792	13.8819		14.0	-.6493	119.1329	2.3604	5.9126		7.0	.2087	4.1043
10.1027	14.8949		15.0	-.6769	131.8521	2.7667	6.8204		8.0	.2145	5.9508
10.7005	15.9053		16.0	-.6988	144.6486	3.1866	7.7412		9.0	.2186	8.2340
11.2755	16.9133		17.0	-.7156	157.5345	3.6183	8.6737		10.0	.2214	10.9782
11.8314	17.9192		18.0	-.7280	170.5485	4.0605	9.6164		11.0	.2235	14.2023
12.3718	18.9235		19.0	-.7371	183.7453	4.5118	10.5680		12.0	.2250	17.9206
12.9005	19.9266		20.0	-.7435	197.1851	4.9707	11.5273		13.0	.2261	22.1434
13.4343	20.9303		22.0	-.7510	225.0144	5.4364	12.4931		14.0	.2269	26.8780
14.9501	23.9320		24.0	-.7545	254.3928	5.9076	13.4645		15.0	.2276	32.1290
15.9572	25.9328		26.0	-.7561	285.5386	6.3836	14.4406		16.0	.2281	37.8990
16.9604	27.9331		28.0	-.7568	318.5694	6.8636	15.4207		17.0	.2285	44.1898
17.9619	29.9333		30.0	-.7571	353.5449	7.3469	16.4041		18.0	.2288	51.0015
$\zeta = 1.02; \omega_n T_1 = 16.0$						7.8331	17.3904		19.0	.2291	58.3337
.5253 <sup>-1</sup>	.1524		.2	-.1359 <sup>-4</sup>	.3995 <sup>-2</sup>	8.3216	18.3789		20.0	.2293	66.1855
.1103	.3095		.4	-.1077 <sup>-3</sup>	.3186 <sup>-1</sup>	9.3042	20.3616		22.0	.2296	83.4430
.1737	.4708		.6	-.3594 <sup>-3</sup>	.1070	10.2922	22.3497		24.0	.2298	102.7627
.2429	.6361		.8	-.8406 <sup>-3</sup>	.2516	11.2840	24.3415		26.0	.2299	124.1329
.3181	.8052		1.0	-.1616 <sup>-2</sup>	.4868	12.2784	26.3359		28.0	.2300	147.5430
.5339	1.2423		1.5	-.5200 <sup>-2</sup>	1.5894	13.2746	28.3321		30.0	.2301	172.9842
.7900	1.6971		2.0	-.1161 <sup>-1</sup>	3.5997	$\zeta = 1.5; \omega_n T_1 = 1.0$					
1.4159	2.6465		3.0	-.3367 <sup>-1</sup>	10.7362	.5243 <sup>-1</sup>	.1525		.2	.2258 <sup>-3</sup>	.2496 <sup>-3</sup>
2.1635	3.6313		4.0	-.6653 <sup>-1</sup>	21.7946	.1094	.3102		.4	.1629 <sup>-2</sup>	.1982 <sup>-2</sup>
2.9887	4.6356		5.0	-.1066	35.8876	.1703	.4725		.6	.4943 <sup>-2</sup>	.6613 <sup>-2</sup>
3.8557	5.6498		6.0	-.1507	52.1135	.2346	.6398		.8	.1051 <sup>-1</sup>	.1544 <sup>-1</sup>
4.7415	6.6683		7.0	-.1967	69.8518	.3016	.8108		1.0	.1839 <sup>-1</sup>	.2960 <sup>-1</sup>
5.6337	7.6886		8.0	-.2431	88.7412	.4774	1.2509		1.5	.4696 <sup>-1</sup>	.9442 <sup>-1</sup>
6.5257	8.7094		9.0	-.2895	108.5830	.6606	1.7027		2.0	.8428 <sup>-1</sup>	.2100
7.4139	9.7302		10.0	-.3355	129.2639	1.0373	2.6208		3.0	.1670	.6245
8.2962	10.7506		11.0	-.3807	150.7110	1.4234	3.5452		4.0	.2437	1.3243
9.1711	11.7705		12.0	-.4251	172.8677	1.8193	4.4729		5.0	.3073	2.3534
10.0372	12.7899		13.0	-.4684	195.6816	2.2257	5.4047		6.0	.3580	3.7512
10.8933	13.8086		14.0	-.5106	219.0971	2.6429	6.3417		7.0	.3975	5.5519
11.7380	14.8266		15.0	-.5514	243.0519	3.0705	7.2846		8.0	.4281	7.7842
12.5695	15.8439		16.0	-.5907	267.4746	3.5079	8.2337		9.0	.4518	10.4718
13.3860	16.8603		17.0	-.6282	292.2822	3.9540	9.1891		10.0	.4701	13.6335
14.1856	17.8758		18.0	-.6637	317.7395	4.4080	10.1502		11.0	.4844	17.2838
14.9658	18.8902		19.0	-.6969	342.6547	4.8690	11.1168		12.0	.4956	21.4337
15.7242	19.9035		20.0	-.7276	367.9898	5.3361	12.0883		13.0	.5044	26.0910
17.1663	21.9262		22.0	-.7800	418.3243	5.8083	13.0640		14.0	.5114	31.2612
18.4972	23.9431		24.0	-.8193	467.4505	6.2851	14.0435		15.0	.5170	36.9476
19.7169	25.9545		26.0	-.8453	514.9228	6.7656	15.0263		16.0	.5214	43.1522
20.8452	27.9611		28.0	-.8605	561.0995	7.2493	16.0118		17.0	.5250	49.8758
21.9122	29.9646		30.0	-.8684	606.9106	7.7358	16.9997		18.0	.5279	57.1184
$\zeta = 1.5; \omega_n T_1 = 0$						8.2245	17.9896		19.0	.5303	64.8793
.5	1.4788	2.1565	2.3554		.6055 <sup>-1</sup>	8.7151	18.9812		20.0	.5322	73.1575
1.0	2.7742	3.7852	4.0222		.4570	9.7009	20.9684		22.0	.5350	91.2614
1.5	3.9937	5.2372	5.4870		1.3250	10.6911	22.9596		24.0	.5369	111.4194
2.0	5.1628	6.5809	6.8362		2.7136	11.6844	24.9535		26.0	.5382	133.6210
2.5	6.2944	7.8469	8.1050		4.6418	12.6798	26.9494		28.0	.5391	157.8570
3.0	7.3991	9.0578	9.3176		7.1112	13.6767	28.9465		30.0	.5396	184.1196
3.5	8.4824	10.2255	10.4863		10.1212	$\zeta = 1.5; \omega_n T_1 = 2.0$					
4.0	9.5492	11.3598	11.6212		13.6696	.5310 <sup>-1</sup>	.1531		.2	.2965 <sup>-4</sup>	.4987 <sup>-3</sup>
4.5	10.6033	12.4683	12.7301		17.7484	.1122	.3125		.4	.2241 <sup>-3</sup>	.3997 <sup>-2</sup>
5.0	11.6466	13.5552	13.8175		22.3612	.1770	.4777		.6	.7114 <sup>-3</sup>	.1318 <sup>-1</sup>
5.5	12.6819	14.6263	14.8887		27.4890	.2468	.6484		.8	.1580 <sup>-2</sup>	.3068 <sup>-1</sup>
6.0	13.7120	15.6865	15.9491		33.1393	.3212	.8240		1.0	.2880 <sup>-2</sup>	.5861 <sup>-1</sup>
6.5	14.7354	16.7335	16.9963		39.3103	.5221	1.2802		1.5	.8093 <sup>-2</sup>	.1846
7.0	15.7552	17.7730	18.0358		45.9928						



TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT  
WITH ZERO - Continued

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_m$	b/B	$R_0 \omega_n^2 / BK$
$\zeta = 1.5; \omega_n T_1 = 2.0$						$\zeta = 1.5; \omega_n T_1 = 8.0$					
0.7375	1.7533		2.0	0.1582 -1	0.4037	7.3854	11.8636		12.0	-0.3804	75.0732
1.1884	2.7245		3.0	.3616 -1	1.1492	8.0230	12.8727		13.0	-.4126	86.1493
1.6489	3.7080		4.0	.5908 -1	2.3214	8.6483	13.8808		14.0	-.4419	97.6521
2.1134	4.6951		5.0	.8163 -1	3.9341	9.2613	14.8880		15.0	-.4683	109.5515
2.5812	5.6836		6.0	.1025	6.0019	9.8622	15.8943		16.0	-.4920	121.8253
3.0523	6.6734		7.0	.1210	8.5386	10.4516	16.8998		17.0	-.5130	134.4581
3.5270	7.6642		8.0	.1372	11.5562	11.0302	17.9046		18.0	-.5314	147.4404
4.0050	8.6562		9.0	.1512	15.0638	11.5987	18.9087		19.0	-.5475	160.7682
4.4862	9.6493		10.0	.1631	19.0684	12.1580	19.9123		20.0	-.5614	174.4423
4.9701	10.6433		11.0	.1731	23.5750	$\zeta = 1.5; \omega_n T_1 = 16.0$					
5.4565	11.6381		12.0	.1816	28.5870	.5371 -1	.1536		.2	-.1265 -4	.3990 -2
5.9450	12.6338		13.0	.1887	34.1066	.1150	.3142		.4	-.9957 -4	.3161 -1
6.4354	13.6301		14.0	.1947	40.1350	.1837	.4811		.6	-.3291 -3	.1051
6.9273	14.6270		15.0	.1996	46.6728	.2599	.6538		.8	-.7601 -3	.2442
7.4206	15.6244		16.0	.2038	53.7202	.3430	.8315		1.0	-.1440 -2	.4654
7.9150	16.6222		17.0	.2072	61.2769	.5788	1.2933		1.5	-.4433 -2	1.4538
8.4103	17.6204		18.0	.2101	69.3424	.8480	1.7725		2.0	-.9446 -2	3.1428
8.9065	18.6189		19.0	.2124	77.9161	1.4608	2.7579		3.0	-.2536 -1	8.6825
9.4032	19.6176		20.0	.2144	86.9975	2.1428	3.7591		4.0	-.4795 -1	16.8844
9.9084	21.6157		22.0	.2174	106.6803	2.8714	4.7664		5.0	-.7562 -1	27.3811
11.3950	23.6144		24.0	.2194	128.3854	3.6319	5.7763		6.0	-.1069	39.8194
12.3927	25.6135		26.0	.2208	152.1080	4.4137	6.7875		7.0	-.1408	53.8951
13.3911	27.6129		28.0	.2217	177.8442	5.2087	7.7991		8.0	-.1762	69.3584
14.3901	29.6125		30.0	.2224	205.5907	6.0110	8.8107		9.0	-.2125	86.0090
$\zeta = 1.5; \omega_n T_1 = 4.0$						6.8159	9.8222		10.0	-.2492	103.6886
.5345 -1	.1534		.2	-.1900 -4	.9972 -3	7.6199	10.8334		11.0	-.2858	122.2709
.1138	.3135		.4	-.1470 -3	.7907 -2	8.4205	11.8442		12.0	-.3220	141.6543
.1807	.4798		.6	-.4775 -3	.2631 -1	9.2156	12.8545		13.0	-.3576	161.7549
.2540	.6518		.8	-.1084 -2	.6118 -1	10.0035	13.8643		14.0	-.3924	182.5011
.3330	.8290		1.0	-.2020 -2	.1167	10.7829	14.8736		15.0	-.4262	203.8297
.5519	1.2900		1.5	-.5977 -2	.3659	11.5527	15.8824		16.0	-.4588	225.6834
.7938	1.7693		2.0	-.1226 -1	.7945	12.3118	16.8907		17.0	-.4901	248.0084
1.3179	2.7567		3.0	-.3058 -1	2.2191	13.0594	17.8984		18.0	-.5201	270.7539
1.8679	3.7600		4.0	-.5391 -1	4.3741	13.7948	18.9056		19.0	-.5487	293.8707
2.4272	4.7682		5.0	-.7949 -1	7.2064	14.5174	19.9123		20.0	-.5756	317.3113
2.9884	5.7776		6.0	-.1054	10.6667	$\zeta = 2.0; \omega_n T_1 = 0$					
3.5477	6.7868		7.0	-.1305	14.7142	.5	1.4819	2.1243	2.2851		.0443
4.1029	7.7953		8.0	-.1538	19.3159	1.0	2.8138	3.8071	3.9862		.3690
4.6531	8.8029		9.0	-.1751	24.4458	1.5	4.0798	5.3431	5.5264		1.1165
5.1980	9.8095		10.0	-.1942	30.0840	2.0	5.3099	6.8051	6.9899		2.3150
5.7374	10.8153		11.0	-.2110	36.2158	2.5	6.5017	8.1884	8.3737		4.0314
6.2717	11.8203		12.0	-.2257	42.8306	3.0	7.6579	9.5009	9.6864		6.2888
6.8013	12.8246		13.0	-.2384	49.9215	3.5	8.7910	10.7675	10.9530		9.0772
7.3267	13.8282		14.0	-.2492	57.4843	4.0	9.9023	11.9901	12.1757		12.4183
7.8482	14.8313		15.0	-.2585	65.5168	4.5	10.9965	13.1782	13.3638		16.3031
8.3665	15.8338		16.0	-.2663	74.0185	5.0	12.0706	14.3269	14.5126		20.7466
8.8819	16.8360		17.0	-.2729	82.9901	5.5	13.1471	15.4800	15.6657		25.6714
9.3948	17.8378		18.0	-.2785	92.4329	6.0	14.2075	16.6005	16.7863		31.1648
9.9056	18.8393		19.0	-.2832	102.3487	6.5	15.2583	17.7021	17.8879		37.1886
10.4147	19.8405		20.0	-.2871	112.7397	7.0	16.3028	18.7911	18.9769		43.7366
11.4285	21.8424		22.0	-.2930	134.9561	7.5	17.3408	19.8673	20.0529		50.8034
12.4380	23.8437		24.0	-.2971	159.0998	8.0	18.3685	20.9228	21.1085		58.4224
13.4446	25.8446		26.0	-.3000	185.1863	8.5	19.4042	21.9941	22.1797		66.5052
14.4491	27.8453		28.0	-.3019	213.2292	9.0	20.4227	23.0311	23.2168		75.1658
15.4522	29.8457		30.0	-.3033	243.2389	9.5	21.4511	24.0878	24.2735		84.2782
$\zeta = 1.5; \omega_n T_1 = 8.0$						10.0	22.4654	25.1160	25.3016		93.9667
.5362 -1	.1535		.2	-.1971 -4	.1994 -2	$\zeta = 2.0; \omega_n T_1 = 0.5$					
.1146	.3139		.4	-.1544 -3	.1581 -1	.5231 -1	.1526		.2	-.1225 -2	.1260 -3
.1827	.4807		.6	-.5073 -3	.5258 -1	.1084	.3104		.4	.8028 -2	.9865 -3
.2579	.6532		.8	-.1165 -2	.1222	.1671	.4726		.6	.2214 -1	.3270 -2
.3396	.8308		1.0	-.2195 -2	.2330	.2274	.6381		.8	.4292 -1	.7580 -2
.5693	1.2927		1.5	-.6667 -2	.7285	.2886	.8057		1.0	.6868 -1	1.4444 -1
.8286	1.7725		2.0	-.1403 -1	1.5773	.4430	1.2288		1.5	.1430	.4559 -1
1.4077	2.7608		3.0	-.3674 -1	4.3735	.5979	1.6518		2.0	.2157	.1014
2.0370	3.7657		4.0	-.6784 -1	8.5432	.9098	2.4912		3.0	.3255	.3086
2.6949	4.7767		5.0	-.1045	13.9274	1.2290	3.3262		4.0	.3890	.6780
3.3683	5.7899		6.0	-.1446	20.3725	1.5591	4.1648		5.0	.4230	1.2518
4.0485	6.8037		7.0	-.1862	27.7439	1.9019	5.0121		6.0	.4408	2.0714
4.7293	7.8173		8.0	-.2280	35.9273	2.2579	5.8705		7.0	.4504	3.1758
5.4061	8.8303		9.0	-.2690	44.8263	2.6269	6.7408		8.0	.4557	4.6013
6.0758	9.8424		10.0	-.2084	54.3601						
6.7361	10.8535		11.0	-.3456	64.4612						

TABLE IV.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER PLANT  
WITH ZERO - Concluded

$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_M$	b/B	$P_0 \omega_n^2 / BK$	$\omega_n T_1$	$\omega_n T_2$	$\omega_n T_3$	$\omega_n T_M$	b/B	$P_0 \omega_n^2 / BK$
$\zeta = 2.0; \omega_n T_1 = 0.5$						$\zeta = 2.0; \omega_n T_1 = 4.0$					
3.0082	7.6229		9.0	0.4587	6.3803	0.5466 -1	0.1547		0.2	-0.3791 -5	0.9943 -3
3.4012	8.5163		10.0	.4604	8.5422	.1184	.3183		.4	-.2909 -4	.7820 -2
3.8050	9.4204		11.0	.4615	11.1123	.1903	.4901		.6	-.9330 -4	.2569 -1
4.2188	10.3344		12.0	.4622	14.1120	.2692	.6688		.8	-.2085 -3	.5880 -1
4.6418	11.2574		13.0	.4627	17.5608	.3538	.8531		1.0	-.3816 -3	.1102
5.0731	12.1888		14.0	.4630	21.4749	.5827	1.3306		1.5	-.1077 -2	.3295
5.5120	13.1278		15.0	.4632	25.8665	.8256	1.8213		2.0	-.2118 -2	.6865
5.9577	14.0735		16.0	.4634	30.7474	1.3267	2.8167		3.0	-.4985 -2	1.8134
6.4096	15.0255		17.0	.4635	36.1269	1.8339	3.8169		4.0	-.8526 -2	3.4781
6.8670	15.9829		18.0	.4636	42.0089	2.3427	4.8178		5.0	-.1242 -1	5.6704
7.3293	16.9453		19.0	.4637	48.4018	2.8518	5.8190		6.0	-.1643 -1	8.3846
7.7960	17.9119		20.0	.4637	55.3059	3.3609	6.8201		7.0	-.2042 -1	11.6161
						3.8697	7.8212		8.0	-.2427 -1	15.3610
						4.3780	8.8222		9.0	-.2793 -1	19.6160
						4.8858	9.8231		10.0	-.3135 -1	24.3781
						5.3930	10.8240		11.0	-.3451 -1	29.6449
						5.8996	11.8248		12.0	-.3740 -1	35.4144
						6.4056	12.8255		13.0	-.4004 -1	41.6849
						6.9110	13.8262		14.0	-.4242 -1	48.4551
						7.4159	14.8268		15.0	-.4456 -1	55.7241
						7.9202	15.8273		16.0	-.4648 -1	63.4910
						8.4241	16.8278		17.0	-.4819 -1	71.7553
						8.9276	17.8282		18.0	-.4971 -1	80.5167
						9.4307	18.8286		19.0	-.5107 -1	89.7748
						9.9334	19.8289		20.0	-.5227 -1	99.5295
$\zeta = 2.0; \omega_n T_1 = 1.0$						$\zeta = 2.0; \omega_n T_1 = 8.0$					
.5360 -1	.1538		.2	.4500 -3	.2498 -3	.5485 -1	.1548		.2	-.1582 -4	.1988 -2
.1137	.3153		.4	.3226 -2	.1961 -2	.1192	.3187		.4	-.1229 -3	.1563 -1
.1788	.4837		.6	.9676 -2	.6464 -2	.1925	.4909		.6	-.3988 -3	.5134 -1
.2477	.6578		.8	.2028 -1	.1486 -1	.2735	.6699		.8	-.9013 -3	.1174
.3189	.8363		1.0	.3488 -1	.2800 -1	.3612	.8545		1.0	-.1668 -2	.2198
.5017	1.2943		1.5	.8521 -1	.8544 -1	.6021	1.3327		1.5	-.4834 -2	.6561
.6854	1.7589		2.0	.1470	.1829	.8634	1.8243		2.0	-.9758 -2	1.3634
1.0496	2.6878		3.0	.2764	.5172	1.4190	2.8223		3.0	-.2416 -1	3.5783
1.4129	3.6089		4.0	.3920	1.0680	1.9997	3.8263		4.0	-.4335 -1	6.8033
1.7795	4.5224		5.0	.4858	1.8706	2.5960	4.8320		5.0	-.6598 -1	10.9719
2.1516	5.4303		6.0	.5574	2.9567	3.2023	5.8384		6.0	-.9097 -1	16.0212
2.5305	6.3353		7.0	.6095	4.3557	3.8146	6.8450		7.0	-.1174	21.8912
2.9171	7.2402		8.0	.6461	6.0954	4.4297	7.8516		8.0	-.1445	28.5253
3.3121	8.1477		9.0	.6712	8.2028	5.0451	8.8581		9.0	-.1718	35.8715
3.7157	9.0598		10.0	.6882	10.7035	5.6587	9.8643		10.0	-.1986	43.8822
4.1280	9.9779		11.0	.6998	13.6210	6.2690	10.8703		11.0	-.2247	52.5152
4.5488	10.9026		12.0	.7077	16.9761	6.8749	11.8759		12.0	-.2498	61.7330
4.9776	11.8341		13.0	.7132	20.7865	7.4754	12.8811		13.0	-.2736	71.5032
5.4138	12.7723		14.0	.7171	25.0674	8.0701	13.8859		14.0	-.2961	81.7978
5.8570	13.7169		15.0	.7199	29.8313	8.6586	14.8903		15.0	-.3171	92.5928
6.3064	14.6674		16.0	.7219	35.0880	9.2408	15.8943		16.0	-.3366	103.8684
6.7615	15.6232		17.0	.7235	40.8455	9.8167	16.8980		17.0	-.3547	115.6083
7.2218	16.5841		18.0	.7246	47.1102	10.3863	17.9014		18.0	-.3712	127.7995
7.6867	17.5494		19.0	.7255	53.8872	10.9499	18.9044		19.0	-.3864	140.4320
8.1557	18.5188		20.0	.7262	61.1792	11.5079	19.9071		20.0	-.4001	153.4983
$\zeta = 2.0; \omega_n T_1 = 2.0$						$\zeta = 2.0; \omega_n T_1 = 16.0$					
.5430 -1	.1544		.2	.8857 -4	.4980 -3	.5494 -1	.1549		.2	-.1164 -4	.3976 -2
.1167	.3174		.4	.6652 -3	.3914 -2	.1197	.3189		.4	-.9095 -4	.3126 -1
.1861	.4882		.6	.2086 -2	.1287 -1	.1936	.4912		.6	-.2968 -3	.1026
.2612	.6658		.8	.4561 -2	.2950 -1	.2758	.6704		.8	-.6745 -3	.2347
.3406	.8488		1.0	.8172 -2	.5535 -1	.3651	.8551		1.0	-.1255 -2	.4392
.5499	1.3219		1.5	.2191 -1	.1665	.6126	1.3331		1.5	-.3687 -2	1.3094
.7656	1.8066		2.0	.4106 -1	.3494	.8847	1.8243		2.0	-.7544 -2	2.7182
1.1975	2.7861		3.0	.8860 -1	.9422	1.4747	2.8212		3.0	-.1919 -1	7.1173
1.6261	3.7668		4.0	.1408	1.8557	2.1068	3.8242		4.0	-.3533 -1	13.4920
2.0542	4.7469		5.0	.1926	3.1159	2.7703	4.8290		5.0	-.5515 -1	21.6771
2.4846	5.7265		6.0	.2419	4.7496	3.4580	5.8347		6.0	-.7791 -1	31.5114
2.9189	6.7062		7.0	.2874	6.7814	4.1645	6.8410		7.0	-.1029	42.8396
3.3577	7.6864		8.0	.3287	9.2329	4.8849	7.8475		8.0	-.1297	55.5164
3.8014	8.6672		9.0	.3658	12.1224	5.6150	8.8541		9.0	-.1576	69.4099
4.2500	9.6489		10.0	.3989	15.4658	6.3513	9.8608		10.0	-.1863	84.4019
4.7034	10.6315		11.0	.4281	19.2762						
5.1613	11.6153		12.0	.4540	23.5645						
5.6234	12.6001		13.0	.4766	28.3398						
6.0895	13.5860		14.0	.4965	33.6092						
6.5592	14.5731		15.0	.5139	39.3787						
7.0321	15.5613		16.0	.5292	45.6528						
7.5080	16.5505		17.0	.5424	52.4351						
7.9867	17.5407		18.0	.5540	59.7281						
8.4677	18.5319		19.0	.5641	67.5339						
8.9510	19.5239		20.0	.5730	75.8536						

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER  
UNSTABLE PLANT WITH ZERO

$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$
$\tau_2/\tau_3 = 0.10; \tau_1/\tau_2 = 0$						$\tau_2/\tau_3 = 0.10; \tau_1/\tau_2 = 2.0$					
0.2936	-1	0.1002	0.1708	0.2	0.3446	1.7798	5.5502	7.0	0.2703	1.7297	
.6008	-1	.2030	.3430	.4	.5733	1.8886	5.9333	7.5	.2947	2.0868	
.9120	-1	.3068	.5156	.6	.8102	1.9946	6.3120	8.0	.3174	2.4852	
.1230		.4120	.6890	.8	.1052	2.0979	6.6864	8.5	.3385	2.9266	
.1555		.5186	.8632	1.0	.2600	2.1987	7.0565	9.0	.3576	3.4125	
.2395		.7917	1.3022	1.5	.1305	2.2970	7.4224	9.5	.3755	3.9444	
.3272		1.0734	1.7462	2.0	.4089	2.3929	7.7843	10.0	.3914	4.5239	
.4182		1.3633	2.1951	2.5	.9880	2.5783	8.4967	11.0	.4184	5.8321	
.5119		1.6608	2.6488	3.0	.2025	2.7556	9.1952	12.0	.4396	7.3504	
.6081		1.9653	3.1072	3.5	.3699	2.9255	9.8813	13.0	.4558	9.0926	
.7062		2.2760	3.5698	4.0	.6212	3.0887	10.5567	14.0	.4680	11.0733	
.8058		2.5924	4.0366	4.5	.9781	3.2456	11.2227	15.0	.4771	13.3069	
.9065		2.9136	4.5070	5.0	.1464	3.3966	11.8804	16.0	.4838	15.8086	
1.0080		3.2390	4.9810	5.5	.2101	3.5422	12.5307	17.0	.4885	18.5924	
1.1099		3.5679	5.4580	6.0	.2916	3.6824	13.1744	18.0	.4920	21.6745	
1.2120		3.8998	5.9378	6.5	.3931	3.8177	13.8121	19.0	.4944	25.0663	
1.3139		4.2341	6.4201	7.0	.5173	3.9460	14.4441	20.0	.4961	28.7834	
1.4156		4.5703	6.9047	7.5	.6665						
1.5167		4.9080	7.3913	8.0	.8432						
1.6172		5.2467	7.8796	8.5	1.0498						
1.7168		5.5862	8.3694	9.0	1.2888						
1.8155		5.9262	8.8606	9.5	1.5625						
1.9133		6.2663	9.3530	10.0	1.8732						
2.0104		6.6061	9.8479	11.0	2.2149						
2.1077		6.9461	10.3407	12.0	2.5318						
2.2052		7.2857	10.8316	13.0	2.8318						
2.3027		7.6243	11.3198	14.0	3.1184						
2.4004		7.9619	11.8048	15.0	3.3964						
2.4981		8.2987	12.2861	16.0	3.6665						
2.5958		8.6346	12.7636	17.0	3.9288						
2.6935		8.9696	13.2374	18.0	4.1832						
2.7912		9.3038	13.7076	19.0	4.4296						
2.8889		9.6373	14.1742	20.0	4.6680						
2.9866		9.9701	14.6374		4.8984						
3.0843		10.3023	15.0971		5.1208						
3.1820		10.6339	15.5534		5.3352						
3.2797		10.9650	16.0064		5.5416						
3.3774		11.2957	16.4564		5.7399						
3.4751		11.6260	16.9034		5.9301						
3.5728		11.9559	17.3474		6.1122						
3.6705		12.2854	17.7884		6.2863						
3.7682		12.6145	18.2264		6.4524						
3.8659		12.9432	18.6614		6.6105						
3.9636		13.2715	19.0934		6.7606						
4.0613		13.5994	19.5228		6.9027						
4.1590		13.9269	19.9494		7.0368						
4.2567		14.2540	20.3731		7.1629						
4.3544		14.5807	20.7939		7.2810						
4.4521		14.9070	21.2119		7.3911						
4.5498		15.2329	21.6271		7.4932						
4.6475		15.5584	22.0395		7.5873						
4.7452		15.8835	22.4491		7.6734						
4.8429		16.2082	22.8559		7.7515						
4.9406		16.5325	23.2599		7.8216						
5.0383		16.8564	23.6619		7.8837						
5.1360		17.1799	24.0619		7.9378						
5.2337		17.5030	24.4599		7.9839						
5.3314		17.8257	24.8549		8.0220						
5.4291		18.1480	25.2469		8.0531						
5.5268		18.4699	25.6359		8.0772						
5.6245		18.7914	26.0219		8.0943						
5.7222		19.1125	26.4049		8.1034						
5.8199		19.4332	26.7849		8.1055						
5.9176		19.7535	27.1619		8.1006						
6.0153		20.0734	27.5359		8.0887						
6.1130		20.3929	27.9069		8.0698						
6.2107		20.7120	28.2749		8.0439						
6.3084		21.0307	28.6389		8.0110						
6.4061		21.3490	28.9989		7.9721						
6.5038		21.6669	29.3549		7.9272						
6.6015		21.9844	29.7069		7.8773						
6.6992		22.3015	30.0549		7.8224						
6.7969		22.6182	30.3989		7.7625						
6.8946		22.9345	30.7399		7.6976						
6.9923		23.2504	31.0779		7.6277						
7.0900		23.5659	31.4119		7.5528						
7.1877		23.8810	31.7419		7.4729						
7.2854		24.1957	32.0679		7.3880						
7.3831		24.5100	32.3909		7.2981						
7.4808		24.8239	32.7099		7.2032						
7.5785		25.1374	33.0249		7.1043						
7.6762		25.4505	33.3359		7.0014						
7.7739		25.7632	33.6429		6.8945						
7.8716		26.0755	33.9459		6.7836						
7.9693		26.3874	34.2449		6.6687						
8.0670		26.6989	34.5399		6.5498						
8.1647		27.0099	34.8309		6.4269						
8.2624		27.3204	35.1179		6.2990						
8.3601		27.6304	35.4009		6.1661						
8.4578		27.9399	35.6799		6.0292						
8.5555		28.2489	35.9549		5.8883						
8.6532		28.5574	36.2259		5.7434						
8.7509		28.8654	36.4929		5.5945						
8.8486		29.1729	36.7559		5.4416						
8.9463		29.4799	37.0149		5.2847						
9.0440		29.7864	37.2699		5.1238						
9.1417		30.0924	37.5209		4.9589						
9.2394		30.3979	37.7679		4.7900						
9.3371		30.7029	38.0109		4.6171						
9.4348		31.0074	38.2499		4.4402						
9.5325		31.3114	38.4849		4.2593						
9.6302		31.6149	38.7159		4.0744						
9.7279		31.9179	38.9429		3.8855						
9.8256		32.2204	39.1659		3.6926						
9.9233		32.5224	39.3849		3.4957						
10.0210		32.8239	39.5999		3.2948						
10.1187		33.1249	39.8109		3.0899						
10.2164		33.4254	40.0179		2.8810						
10.3141		33.7254	40.2209		2.6681						
10.4118		34.0249	40.4199		2.4512						
10.5095		34.3239	40.6149		2.2303						
10.6072		34.6224	40.8059		2.0054						
10.7049		34.9204	40.9929		1.7765						
10.8026		35.2179	41.1759		1.5436						
10.9003		35.5149	41.3549		1.3067						
11.0000		35.8114	41.5299		1.0658						
11.0997		36.1074	41.7009		0.8209						
11.1994		36.4029	41.8679		0.5720						
11.2991		36.6979	42.0309		0.3191						
11.3988		36.9924	42.1899		0.0622						
11.4985		37.2864	42.3449								
11.5982		37.5799	42.4959								
11.6979		37.8729	42.6429								
11.7976		38.1654	42.7859								
11.8973		38.4574	42.9249								
11.9970		38.7489	43.0599								
12.0967		39.0399	43.1909								
12.1964		39.3304	43.3179								
12.2961		39.6204	43.4409								
12.3958		39.9099	43.5599								
12.4955		40.1989	43.6749								
12.5952		40.4874	43.7859								
12.6949		40.7754	43.8929								
12.7946		41.0629	43.9959								
12.8943		41.3504	44.0949								
12.9940		41.6374	44.1899								
13.0937		41.9239	44.2809								
13.1934		42.2099	44.3679								

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER  
UNSTABLE PLANT WITH ZERO - Continued

$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$
$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 0$						$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 3.0$					
0.1537	0.5155	0.8619	1.0		0.6484 -3	0.5050 -1	0.1505		0.2	0.3134 -4	0.1875 -3
.2352	.7846	1.2995	1.5		.3261 -2	.1019	.3022		.4	.2423 -3	.1498 -2
.3191	1.0608	1.7418	2.0		.1022 -1	.1540	.4552		.6	.7901 -3	.5048 -2
.4047	1.3436	2.1889	2.5		.2469 -1	.2067	.6094		.8	.1808 -2	.1194 -1
.4917	1.6324	2.6408	3.0		.5057 -1	.2599	.7650		1.0	.3409 -2	.2324 -1
.5792	1.9266	3.0973	3.5		.9234 -1	.3936	1.1593		1.5	.1050 -1	.7761 -1
.6669	2.2253	3.5584	4.0		.1550	.5268	1.5608		2.0	.2265 -1	.1813
.7542	2.5280	4.0238	4.5		.2438	.6576	1.9678		2.5	.4013 -1	.3475
.8406	2.8337	4.4931	5.0		.3645	.7843	2.3785		3.0	.6275 -1	.5876
.9259	3.1420	4.9661	5.5		.5226	.9061	2.7910		3.5	.8997 -1	.9106
1.0095	3.4520	5.4426	6.0		.7240	1.0220	3.2036		4.0	.1211	1.3238
1.0912	3.7633	5.9221	6.5		.9744	1.1318	3.6145		4.5	.1552	1.8326
1.1708	4.0751	6.4043	7.0		1.2793	1.2352	4.0224		5.0	.1915	2.4412
1.2481	4.3872	6.8890	7.5		1.6443	1.3323	4.4260		5.5	.2291	3.1526
1.3230	4.6989	7.3799	8.0		2.0746	1.4233	4.8243		6.0	.2673	3.9689
1.3953	5.0100	7.8647	8.5		2.5752	1.5084	5.2164		6.5	.3053	4.8912
1.4650	5.3202	8.3552	9.0		3.1509	1.5879	5.6018		7.0	.3426	5.9199
1.5320	5.6291	8.8471	9.5		3.8062	1.6619	5.9798		7.5	.3786	7.0547
1.5963	5.9366	9.3403	10.0		4.5455	1.7310	6.3502		8.0	.4128	8.2947
1.7170	6.5467	10.3297	11.0		6.2917	1.7952	6.7127		8.5	.4450	9.6388
1.8271	7.1495	11.3223	12.0		8.4184	1.8551	7.0671		9.0	.4747	11.0854
1.9273	7.7445	12.3172	13.0		10.9509	1.9109	7.4135		9.5	.5018	12.6330
2.0179	8.3317	13.3137	14.0		13.9106	1.9628	7.7521		10.0	.5262	14.2801
2.0988	8.9111	14.3114	15.0		17.3160	2.0563	8.4070		11.0	.5671	17.8693
2.1734	9.4832	15.3098	16.0		21.1825	2.1380	9.0349		12.0	.5980	21.8503
2.2395	10.0483	16.3088	17.0		25.5232	2.2095	9.6401		13.0	.6204	26.2292
2.2988	10.6069	17.3091	18.0		30.3487	2.2725	10.2266		14.0	.6360	31.0194
$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 0.5$						$\tau_2/\tau_3 = 0.25; \tau_1/\tau_2 = 5.0$					
.4857 -1	.1483	.2	-0.9237 -3	.3128 -4	.5066 -1	.1507	.2	.1764 -4	.3124 -3		
.9475 -1	.2932	.4	-.6106 -2	.2510 -3	.1026	.3029	.4	.1382 -3	.2496 -2		
.1391	.4348	.6	-.1715 -1	.8519 -3	.1555	.4567	.6	.4564 -3	.8410 -2		
.1821	.5736	.8	-.3405 -1	.2035 -2	.2095	.6121	.8	.1058 -2	.1989 -1		
.2240	.7100	1.0	-.5611 -1	.4016 -2	.2641	.7692	1.0	.2020 -2	.3871 -1		
.3257	1.0438	1.5	-.1276	.1400 -1	.4031	1.1692	1.5	.6420 -2	.1292		
.4245	1.3716	2.0	-.2116	.3447 -1	.5434	1.5791	2.0	.1427 -1	.3013		
.5213	1.6966	2.5	-.2968	.7008 -1	.6826	1.9977	2.5	.2605 -1	.5769		
.6165	2.0206	3.0	-.3836	.1260	.8186	2.4234	3.0	.4191 -1	.9736		
.7100	2.3442	3.5	-.4630	.2076	.9500	2.8545	3.5	.6177 -1	1.5050		
.8017	2.6678	4.0	-.5356	.3206	1.0796	3.2890	4.0	.8538 -1	2.1813		
.8913	2.9911	4.5	-.6008	.4708	1.1946	3.7255	4.5	.1123	3.0095		
.9787	3.3140	5.0	-.6586	.6640	1.3066	4.1622	5.0	.1422	3.9941		
1.0636	3.6362	5.5	-.7094	.9058	1.4116	4.5978	5.5	.1745	5.1378		
1.1459	3.9575	6.0	-.7537	1.2013	1.5096	5.0313	6.0	.2087	6.4419		
1.2256	4.2776	6.5	-.7920	1.5573	1.6008	5.4616	6.5	.2443	7.9067		
1.3024	4.5962	7.0	-.8251	1.9776	1.6855	5.8879	7.0	.2810	9.5315		
1.3765	4.9131	7.5	-.8533	2.4675	1.7640	6.3095	7.5	.3182	11.3147		
1.4476	5.2283	8.0	-.8774	3.0318	1.8367	6.7258	8.0	.3556	13.2540		
1.5159	5.5415	8.5	-.8979	3.6748	1.9039	7.1362	8.5	.3929	15.3462		
1.5814	5.8526	9.0	-.9151	4.4009	1.9660	7.5402	9.0	.4297	17.5871		
1.6440	6.1616	9.5	-.9297	5.2141	2.0234	7.9373	9.5	.4656	19.9715		
1.7039	6.4684	10.0	-.9419	6.1178	2.0762	8.3269	10.0	.5003	22.4935		
1.8156	7.0754	11.0	-.9607	8.2111	2.1698	9.0825	11.0	.5651	27.9217		
$\tau_2/\tau_3 = 0.5; \tau_1/\tau_2 = 0$						$\tau_2/\tau_3 = 0.5; \tau_1/\tau_2 = 0.5$					
.2949 -1	.1004	0.1709	.2	.2192 -3	.2949 -1	.1004	0.1709	.2	.2192 -3		
.5935 -1	.2017	.3423	.4	.3336 -4	.5935 -1	.2017	.3423	.4	.3336 -4		
.8953 -1	.3037	.5142	.6	.1683 -3	.8953 -1	.3037	.5142	.6	.1683 -3		
.1200	.4066	.6866	.8	.5316 -3	.1200	.4066	.6866	.8	.5316 -3		
.1507	.5103	.8997	1.0	.1295 -2	.1507	.5103	.8997	1.0	.1295 -2		
.2280	.7729	1.2950	1.5	.6512 -2	.2280	.7729	1.2950	1.5	.6512 -2		
.3055	1.0401	1.7346	2.0	.2039 -1	.3055	1.0401	1.7346	2.0	.2039 -1		
.3825	1.3114	2.1789	2.5	.4919 -1	.3825	1.3114	2.1789	2.5	.4919 -1		
.4582	1.5862	2.6281	3.0	.1005	.4582	1.5862	2.6281	3.0	.1005		
.5318	1.8639	3.0821	3.5	.1831	.5318	1.8639	3.0821	3.5	.1831		
.6029	2.1440	3.5411	4.0	.3063	.6029	2.1440	3.5411	4.0	.3063		
.6708	2.4256	4.0048	4.5	.4799	.6708	2.4256	4.0048	4.5	.4799		
.7352	2.7083	4.4731	5.0	.7140	.7352	2.7083	4.4731	5.0	.7140		
.7959	2.9914	4.9455	5.5	1.0181	.7959	2.9914	4.9455	5.5	1.0181		
.8525	3.2744	5.4218	6.0	1.4017	.8525	3.2744	5.4218	6.0	1.4017		
.9051	3.5568	5.9017	6.5	1.8733	.9051	3.5568	5.9017	6.5	1.8733		
.9536	3.8383	6.3846	7.0	2.4407	.9536	3.8383	6.3846	7.0	2.4407		
.9982	4.1185	6.8703	7.5	3.1108	.9982	4.1185	6.8703	7.5	3.1108		
1.0388	4.3973	7.3584	8.0	3.8898	1.0388	4.3973	7.3584	8.0	3.8898		
1.0758	4.6744	7.8486	8.5	4.7828	1.0758	4.6744	7.8486	8.5	4.7828		
1.1093	4.9498	8.3405	9.0	5.7941	1.1093	4.9498	8.3405	9.0	5.7941		
1.1395	5.2234	8.8339	9.5	6.9271	1.1395	5.2234	8.8339	9.5	6.9271		
1.1667	5.4952	9.3285	10.0	8.1847	1.1667	5.4952	9.3285	10.0	8.1847		
1.2130	6.0336	10.3206	11.0	11.0820	1.2130	6.0336	10.3206	11.0	11.0820		
1.2500	6.5655	11.3155	12.0	14.4978	1.2500	6.5655	11.3155	12.0	14.4978		
1.2793	7.0916	12.3122	13.0	18.4381	1.2793	7.0916	12.3122	13.0	18.4381		
1.3025	7.6127	13.3102	14.0	22.9050	1.3025	7.6127	13.3102	14.0	22.9050		
1.3207	8.1297	14.3089	15.0	27.8983	1.3207	8.1297	14.3089	15.0	27.8983		
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 0.5$						$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 0.5$					
.4827 -1	.1480	.2	-.1026 -2	.6256 -4	.4827 -1	.1480	.2	-.1026 -2	.6256 -4		
.9360 -1	.2919	.4	-.6780 -2	.5018 -3	.9360 -1	.2919	.4	-.6780 -2	.5018 -3		
.1366	.4319	.6	-.1902 -1	.1703 -2	.1366	.4319	.6	-.1902 -1	.1703 -2		
.1778	.5683	.8	-.3774 -1	.4066 -2	.1778	.5683	.8	-.3774 -1	.4066 -2		

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER  
UNSTABLE PLANT WITH ZERO - Continued

$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_M/\tau_2$	b/B	$R_0/BK\tau_2^2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_M/\tau_2$	b/B	$R_0/BK\tau_2^2$		
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 0.5$						$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 5.0$							
0.2174	0.7019	1.0	-0.6213	-1	0.8019	-2	0.8960	3.2148	4.0	0.1275	4.2061		
.3115	1.0263	1.5	-.1409		.2793	-1	.9685	3.6328	4.5	.1657	5.7370		
.3999	1.3416	2.0	-.2330		.6868	-1	1.0313	4.0486	5.0	.2069	7.5184		
.4835	1.6514	2.5	-.3281		.1393		1.0852	4.4607	5.5	.2502	9.5402		
.5625	1.9576	3.0	-.4198		.2498		1.1311	4.8677	6.0	.2946	11.7899		
.6370	2.2608	3.5	-.5048		.4101		1.1701	5.2688	6.5	.3395	14.2534		
.7068	2.5614	4.0	-.5814		.6304		1.2031	5.6629	7.0	.3839	16.9152		
.7719	2.8596	4.5	-.6492		.9207		1.2309	6.0495	7.5	.4275	19.7592		
.8322	3.1551	5.0	-.7081		1.2902		1.2543	6.4279	8.0	.4694	22.7689		
.8878	3.4481	5.5	-.7588		1.7474		1.2740	6.7976	8.5	.5094	25.9273		
.9389	3.7384	6.0	-.8018		2.2999		1.2905	7.1581	9.0	.5470	29.2180		
.9855	4.0260	6.5	-.8381		2.9546		1.3045	7.5091	9.5	.5819	32.6249		
1.0279	4.3108	7.0	-.8683		3.7174								
1.0663	4.5930	7.5	-.8934		4.5932								
1.1010	4.8725	8.0	-.9140		5.5862								
1.1323	5.1496	8.5	-.9309		6.6999								
1.1604	5.4242	9.0	-.9447		7.9372								
1.1856	5.6966	9.5	-.9558		9.3001								
1.2081	5.9669	10.0	-.9648		10.7906								
1.2461	6.5017	11.0	-.9778		14.1585								
1.2764	7.0298	12.0	-.9862		18.0473								
1.3002	7.5523	13.0	-.9914		22.4608								
1.3190	8.0703	14.0	-.9947		27.3286								
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 2.0$						$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 0$							
.4997	1.500	.2	.5944	-4	.2499	-3	.2927	1.000	0.1707	.2	.0408	-4	
.9975	3.002	.4	.4521	-3	.1997	-2	.5756	2.000	0.4444	.4	.6660	-4	
.1492	4.506	.6	.2450	-2	.6729	-2	.8222	2.750	0.4918	.5	1.8360	-4	
.1981	6.013	.8	.3265	-2	.1591	-1	.6042	3.000	0.3958	.6	2.4793	-4	
.2463	7.523	1.0	.6054	-2	.3098	-1	.6223	3.250	0.8777	6.5	3.2474	-4	
.3629	1.1308	1.5	.1790	-1	.1033		.6369	3.500	0.3631	7.0	4.1447	-4	
.4725	1.5096	2.0	.3706	-1	.2409		.6487	3.750	0.8513	7.5	5.1739	-4	
.5739	1.8869	2.5	.6299	-1	.4609		.6581	4.000	7.3419	8.0	6.3366	-4	
.6665	2.2610	3.0	.9447	-1	.7768		.6656	4.250	7.8344	8.5	7.6335	-4	
.7502	2.6301	3.5	.1299		1.1991		.6715	4.500	8.3289	9.0	9.0644	-4	
.8251	2.9926	4.0	.1675		1.7350		.6762	4.750	8.8288	9.5	10.6291	-4	
.8917	3.3475	4.5	.2057		2.3890		.6799	5.000	9.3201	10.0	12.3266	-4	
.9507	3.6939	5.0	.2432		3.1637		.6851	5.500	10.3149	11.0	16.1173	-4	
1.0027	4.0315	5.5	.2788		4.0595		.6882	6.000	11.3118	12.0	20.4300	-4	
1.0484	4.3603	6.0	.3118		5.0763		.6902	6.500	12.3098	13.0	25.2587	-4	
1.0886	4.6803	6.5	.3416		6.2131		.6913	7.000	13.3087	14.0	30.5987	-4	
1.1239	4.9920	7.0	.3681		7.4688		.6920	7.500	14.3080	15.0	36.4466	-4	
1.1549	5.2960	7.5	.3911		8.8424		.6925	8.000	15.3073	16.0	42.7998	-4	
1.1821	5.5929	8.0	.4108		10.3334		.6927	8.500	16.3075	17.0	49.6566	-4	
1.2060	5.8835	8.5	.4275		11.9415		.6929	9.000	17.3071	18.0	57.0158	-4	
1.2271	6.1685	9.0	.4414		13.6669		.6930	9.500	18.3070	19.0	64.8766	-4	
1.2456	6.4485	9.5	.4529		15.5104		.6931	10.000	19.3069	20.0	73.2384	-4	
1.2619	6.7243	10.0	.4623		17.4727								
1.2891	7.2653	11.0	.4762		21.7577		.4768	1.474	.2	-.1231	-2	-.1250	-3
1.3103	7.7954	12.0	.4851		26.5308		.9131	2.893	.4	-.8118	-2	-.1002	-2
1.3270	8.3177	13.0	.4907		31.7991		.1316	4.259	.6	-.2272	-1	.3396	-2
							.1692	5.579	.8	-.4496	-1	.8006	-2
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 3.0$						$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 0.5$							
.5017	1.502	.2	.4477	-4	.3749	-3	.2044	6.899	1.0	-.7379	-1	.1594	-2
1.006	3.011	.4	.3461	-3	.2996	-2	.2386	8.219	1.5	-.1659		.5517	-1
1.509	4.526	.6	.1128	-2	1.009	-1	.2728	9.539	2.0	-.2719		.1346	-1
.8011	6.050	.8	.2580	-2	.2385	-1	.3070	1.0871	2.5	-.3789		.2704	
.2509	7.582	1.0	.4860	-2	.4642	-1	.3412	1.3830	3.0	-.4796		.4784	
.3724	1.1448	1.5	.1493	-1	.1547		.3754	1.6790	3.5	-.5700		.7738	
.4878	1.5399	2.0	.3207	-1	.3599		.4096	1.9750	4.0	-.6689		1.1693	
.9952	1.9299	2.5	.5647	-1	.6865		.4438	2.2710	4.5	-.7659		1.6759	
.6933	2.3247	3.0	.8799	-1	1.1531		.4780	2.5670	5.0	-.8609		2.3013	
.7817	2.7103	3.5	.1244		1.7724		.5122	2.8630	5.5	-.9555		3.0519	
.8604	3.1065	4.0	.1655		2.5517		.5464	3.1590	6.0	-.8555		3.9316	
.9297	3.4937	4.5	.2094		3.4940		.5806	3.4550	6.5	-.8857		4.9428	
.9943	3.8723	5.0	.2547		4.5982		.6148	3.7510	7.0	-.9099		6.0869	
1.0430	4.2430	5.5	.3000		5.8606		.6490	4.0470	7.5	-.9292		7.3642	
1.0886	4.6051	6.0	.3444		7.2755		.6832	4.3430	8.0	-.9444		8.7747	
1.1280	4.9579	6.5	.3866		8.8354		.7174	4.6390	8.5	-.9565		10.3178	
1.1619	5.3010	7.0	.4261		10.5324		.7516	4.9350	9.0	-.9659		11.9931	
1.1910	5.6345	7.5	.4629		12.3585		.7858	5.2310	9.5	-.9734		13.7995	
1.2162	5.9584	8.0	.4969		14.3061		.8200	5.5270	10.0	-.9792		15.7366	
1.2378	6.2732	8.5	.5286		16.3688		.8542	5.8230	11.0	-.9873		19.9991	
1.2564	6.5794	9.0	.5487		18.4515		.8884	6.1190	12.0	-.9923		24.7754	
1.2726	6.8776	9.5	.5701		20.8205		.9226	6.4150	13.0	-.9953		30.0610	
1.2865	7.1689	10.0	.5883		23.2038								
1.3092	7.4331	11.0	.6159		28.2815								
$\tau_2/\tau_3 = 0.50; \tau_1/\tau_2 = 5.0$						$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 2.0$							
.4933	1.494	.2	.8913	-4	.4997	-3	.9714	2.978	.4	.6773	-3	.3990	-2
.9714	2.978	.4	.6773	-3	.3990	-2	.1432	4.454	.6	.2169	-2	.1343	-1
.1432	4.454	.6	.2169	-2	.1343	-1	.1873	5.922	.8	.4871	-2	.3168	-1
.1873	5.922	.8	.4871	-2	.3168	-1	.2293	7.383	1.0	.9003	-2	.6150	-1
.2293	7.383	1.0	.9003	-2	.6150	-1	.3242	1.1006	1.5	.2632	-1	.2032	
.3242	1.1006	1.5	.2632	-1	.2032		.4042	1.4578	2.0	.5360	-1	.4669	
.4042	1.4578	2.0	.5360	-1	.4669		.4696	1.8089	2.5	.8924	-1	.8767	
.4696	1.8089	2.5	.8924	-1	.8767		.5217	2.1523	3.0	.1306		1.4456	
.5217	2.1523	3.0	.1306		1.4456		.5625	2.4972	3.5	.1747		2.2771	
.5625	2.4972	3.5	.1747		2.2771		.5938	2.8421	4.0	.2189		3.0684	
.5938	2.8421	4.0	.2189		3.0684		.6251	3.1870	4.5	.2612		4.1122	
.6251	3.1870	4.5	.2612		4.1122								

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER  
UNSTABLE PLANT WITH ZERO - Continued

$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$	$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$		
$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 2.0$						$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 0.5$							
0.6358	3.4359		5.0	0.3001	5.3000	0.1793	0.6555		1.0	-0.9524	-1	0.3106	-1
.6495	3.7342		5.5	.3347	6.6233	.2326	.9305		1.5	-.2084		.1049	
.6598	4.0247		6.0	.3648	8.0747	.2703	1.1873		2.0	-.3320		.2483	
.6677	4.3081		6.5	.3905	9.6487	.2965	1.4340		2.5	-.4500		.4816	
.6736	4.5855		7.0	.4119	11.3413	.3142	1.6755		3.0	-.5547		.8207	
.6781	4.8577		7.5	.4296	13.1500	.3259	1.9150		3.5	-.6436		1.2768	
.6816	5.1256		8.0	.4440	15.0736	.3335	2.1543		4.0	-.7170		1.8571	
.6842	5.3899		8.5	.4557	17.1118	.3384	2.3943		4.5	-.7765		2.5656	
.6862	5.6513		9.0	.4651	19.2652	.3415	2.6354		5.0	-.8242		3.4044	
.6878	5.9103		9.5	.4725	21.5341	.3434	2.8779		5.5	-.8621		4.3742	
.6890	6.1674		10.0	.4784	23.9198	.3446	3.1216		6.0	-.8921		5.4750	
.6906	6.4774		11.0	.4868	29.0453	.3454	3.3665		6.5	-.9156		6.7064	
						.3458	3.6123		7.0	-.9341		8.0679	
						.3461	3.8590		7.5	-.9486		9.5588	
						.3463	4.1063		8.0	-.9599		11.1784	
						.3464	4.3543		8.5	-.9687		12.9263	
						.3465	4.6026		9.0	-.9756		14.8021	
$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 3.0$						$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 2.0$							
.4955	1.496		.2	.7160	7.494	.4805	1.482		.2	.1484	-3	.9983	-3
.9790	1.790		.4	.5531	9.595	.4918	1.791		.4	.1123	-2	.7943	-2
.1448	1.4475		.6	.1800	2.013	.1315	.4351		.6	.3573	-2	.2657	-1
.1900	.5962		.8	.4109	4.750	.1666	.5746		.8	.7954	-1	.6217	-1
.2333	.7449		1.0	.7716	9.220	.1973	.7118		1.0	.1454	-1	.1194	
.3317	1.1169		1.5	.2345	3.042	.2562	1.0471		1.5	.4093	-1	.3807	
.4149	1.4893		2.0	.4958	1.6978	.2939	1.3734		2.0	.7930	-1	.8362	
.4827	1.8610		2.5	.8551	1.3059	.3166	1.6923		2.5	1.257		1.4927	
.5361	2.2303		3.0	1.2614	2.1432	.3298	2.0046		3.0	1.748		2.3383	
.5770	2.5952		3.5	1.788	3.2088	.3372	2.3101		3.5	2.229		3.3544	
.6078	2.9540		4.0	.2308	4.4906	.3413	2.6091		4.0	2.678		4.5224	
.6305	3.3055		4.5	.2833	5.9705	.3436	2.9015		4.5	3.079		5.8271	
.6472	3.6485		5.0	.3342	7.6281	.3449	3.1877		5.0	3.429		7.2572	
.6594	3.9825		5.5	.3821	9.4432	.3456	3.4682		5.5	3.726		8.8052	
.6682	4.3073		6.0	.4261	11.3977	.3460	3.7436		6.0	3.975		10.4662	
.6746	4.6230		6.5	.4655	13.4763	.3463	4.0143		6.5	4.181		12.2376	
.6794	4.9297		7.0	.5003	15.6669	.3464	4.2813		7.0	4.349		14.1186	
.6828	5.2282		7.5	.5303	17.9603	.3465	4.5449		7.5	4.484		16.1092	
.6854	5.5190		8.0	.5558	20.3509				8.0				
.6872	5.8030		8.5	.5772	22.8353								
.6887	6.0810		9.0	.5949	25.4125								
.6897	6.3539		9.5	.6095	28.0833								
.6905	6.6224		10.0	.6213	30.8496								
$\tau_2/\tau_3 = 1.0; \tau_1/\tau_2 = 5.0$						$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 3.0$							
.4969	1.498		.2	.4702	1.249	.4824	1.484		.2	.1252	-3	.1497	-2
.9853	1.793		.4	.3680	.9975	.9265	1.794		.4	.9633	-2	.1192	-1
.1462	1.4492		.6	.1214	.3355	.1329	.4376		.6	.3116	-2	.3985	-1
.1923	.5994		.8	.2807	.7916	.1687	.5793		.8	.7052	-2	.9326	-1
.2367	.7501		1.0	.5341	1.536	.2002	.7198		1.0	.1310	-1	.1791	
.3384	1.1303		1.5	.1677	.5069	.2604	1.0679		1.5	.3836	-1	.5702	
.4247	1.5162		2.0	.3658	1.1618	.2984	1.4144		2.0	.7730	-1	1.2480	
.4949	1.9075		2.5	.6505	2.1705	.3207	1.7603		2.5	1.264		2.2151	
.5496	2.3029		3.0	.1013	3.5516	.3307	2.1049		3.0	.1813		3.4438	
.5908	2.7004		3.5	.1438	5.2954	.3330	2.4466		3.5	.2381		4.8968	
.6209	3.0978		4.0	.1908	7.3734	.3395	2.7836		4.0	.2938		6.5391	
.6425	3.4932		4.5	.2403	9.7484	.3429	3.1145		4.5	.3466		8.3415	
.6578	3.8849		5.0	.2908	12.3821	.3446	3.4383		5.0	.3952		10.2814	
.6685	4.2714		5.5	.3412	15.2387	.3456	3.7544		5.5	.4389		12.3416	
.6759	4.6517		6.0	.3903	18.2865	.3460	4.0626		6.0	.4776		14.5094	
.6810	5.0249		6.5	.4375	21.4981	.3463	4.3631		6.5	.5111		16.7756	
.6846	5.3903		7.0	.4823	24.8495	.3464	4.6562		7.0	.5398		19.1344	
.6871	5.7472		7.5	.5241	28.3197								
.6888	6.0953		8.0	.5626	31.8906								
$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 0$						$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 0$							
.2884	.9917	0.1703	.2		.0823	.4840	1.486		.2	.8611	-4	.2496	-2
.5668	1.967	.3400	.4		.1329	.9321	2.949		.4	.6716	-3	.1986	-1
.8335	.2926	.5093	.6		.6698	.1340	.4395		.6	.2202	-2	.6644	-1
.1087	.3870	.6783	.8		.2104	.1705	.5831		.8	.5051	-2	.1555	
.1327	.4799	.8473	1.0		.5098	.2026	.7264		1.0	.9509	-2	.2987	
.1860	.7069	1.2709	1.5		.2513	.2641	1.0860		1.5	.2878	-1	.9506	
.2294	.9280	1.6986	2.0		.7656	.3024	1.4520		2.0	.5981	-1	2.0766	
.2635	1.1459	2.1324	2.5		.1785	.3242	1.8258		2.5	.1006		3.6708	
.2891	1.3628	2.5737	3.0		.3504	.3356	2.2062		3.0	.1482		5.6741	
.3077	1.5809	3.0232	3.5		.6102	.3413	2.5903		3.5	.1996		8.0141	
.3208	1.8014	3.4806	4.0		.9725	.3441	2.9753		4.0	.2525		10.6271	
.3298	2.0251	3.9454	4.5		1.4485	.3454	3.3585		4.5	.3053		13.4641	
.3358	2.2524	4.4166	5.0		2.0462	.3460	3.7381		5.0	.3568		16.4884	
.3397	2.4832	4.8935	5.5		2.7705	.3463	4.1125		5.5	.4065		19.6720	
.3422	2.7173	5.3750	6.0		3.6245	.3464	4.4804		6.0	.4536		22.9924	
.3439	2.9542	5.8604	6.5		4.6095	.3465	4.8408		6.5	.4977		26.4307	
.3449	3.1937	6.3488	7.0		5.7263								
.3455	3.4352	6.8396	7.5		6.9746								
.3459	3.6784	7.3325	8.0		8.3541								
.3462	3.9230	7.8269	8.5		9.8643								
.3463	4.1688	8.3225	9.0		11.5044								
.3464	4.4155	8.8190	9.5		13.2740								
.3465	4.6628	9.3163	10.0		15.1723								
$\tau_2/\tau_3 = 2.0; \tau_1/\tau_2 = 0.5$						$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 0$							
.4648	1.461		.2	-.1638	-.2497	.2754	1.469		.2			.2063	-4
.6675	1.2841		.4	-.1075	-.1994	.5135	1.870		.4			.3272	-3
.1218	.4143		.6	-.2988	-.6720	.7131	2.2716		.6			.1619	-2
.1525	.5378		.8	-.5861	-.1591	.8757	2.6640		.8			.4965	-2
						.1005	3.0516		1.0			.1169	-1
						.1212	3.4275		1.5			.5295	-1
						.1312	3.8004		2.0			.1474	
						.1357	4.1664		2.5			.3153	
						.1375	4.5360		3.0			.5730	
						.1382	4.9095		3.5			.9330	
						.1385	5.2817		4.0			1.4049	
						.1385	5.6500		4.5			1.9955	

TABLE V.- NORMALIZED OPTIMUM RESPONSES FOR A TYPE 2 FOURTH-ORDER  
UNSTABLE PLANT WITH ZERO - Concluded

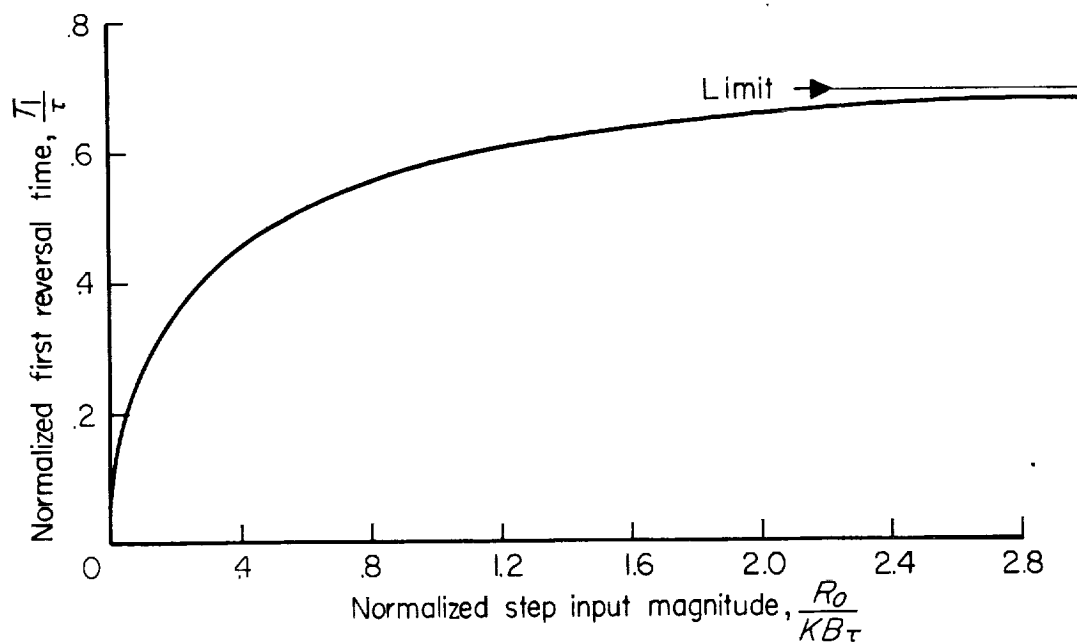
$T_1/\tau_2$	$T_2/\tau_2$	$T_3/\tau_2$	$T_m/\tau_2$	$b/B$	$R_0/BK\tau_2^2$
$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 0.5$					
0.4295 <sup>-1</sup>	0.1422		0.2	-0.2838 <sup>-2</sup>	0.6188 <sup>-3</sup>
.7370 <sup>-1</sup>	.2692		.4	-.1813 <sup>-1</sup>	.4824 <sup>-2</sup>
.9509 <sup>-1</sup>	.3830		.6	-.4851 <sup>-1</sup>	.1570 <sup>-1</sup>
.1097	.4869		.8	-.9117 <sup>-1</sup>	.3572 <sup>-1</sup>
.1195	.5840		1.0	-.1419	.6679 <sup>-1</sup>
.1320	.8116		1.5	-.2813	.2024
.1364	1.0327		2.0	-.4147	.4338
.1379	1.2553		2.5	-.5303	.7728
.1384	1.4818		3.0	-.6264	1.2273
.1386	1.7124		3.5	-.7044	1.8029
$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 2.0$					
.4428 <sup>-1</sup>	.1446		.2	.3236 <sup>-3</sup>	.2475 <sup>-2</sup>
.7729 <sup>-1</sup>	.2797		.4	.2389 <sup>-2</sup>	.1922 <sup>-1</sup>
.1003	.4076		.6	.7313 <sup>-2</sup>	.6192 <sup>-1</sup>
.1155	.5310		.8	.1552 <sup>-1</sup>	.1382
.1251	.6519		1.0	.2689 <sup>-1</sup>	.2519
.1353	.9514		1.5	.6612 <sup>-1</sup>	.7009
.1379	1.2522		2.0	.1143	1.3654
.1385	1.5536		2.5	.1651	2.2166
.1386	1.8532		3.0	.2146	3.2311
$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 3.0$					
.4444 <sup>-1</sup>	.1449		.2	.2837 <sup>-3</sup>	.3713 <sup>-2</sup>
.7775 <sup>-1</sup>	.2809		.4	.2130 <sup>-2</sup>	.2884 <sup>-1</sup>
.1011	.4110		.6	.6634 <sup>-2</sup>	.9293 <sup>-1</sup>
.1164	.5378		.8	.1431 <sup>-1</sup>	.2074
.1259	.6637		1.0	.2520 <sup>-1</sup>	.3778
.1358	.9822		1.5	.6426 <sup>-1</sup>	1.0462
.1381	1.3102		2.0	.1147	2.0235
.1385	1.6447		2.5	.1708	3.2571
.1386	1.9812		3.0	.2284	4.7049
$\tau_2/\tau_3 = 5.0; \tau_1/\tau_2 = 5.0$					
.4457 <sup>-1</sup>	.1451		.2	.2018 <sup>-3</sup>	.6188 <sup>-2</sup>
.7813 <sup>-1</sup>	.2820		.4	.1537 <sup>-2</sup>	.4809 <sup>-1</sup>
.1017	.4138		.6	.4853 <sup>-2</sup>	.1550
.1171	.5436		.8	.1062 <sup>-1</sup>	.3461
.1266	.6739		1.0	.1894 <sup>-1</sup>	.6300
.1362	1.0106		1.5	.4977 <sup>-1</sup>	1.7398
.1382	1.3664		2.0	.9128 <sup>-1</sup>	3.3468
.1386	1.7368		2.5	.1393	5.3515

TABLE VI.- REFERENCES FOR OPTIMUM SYSTEMS

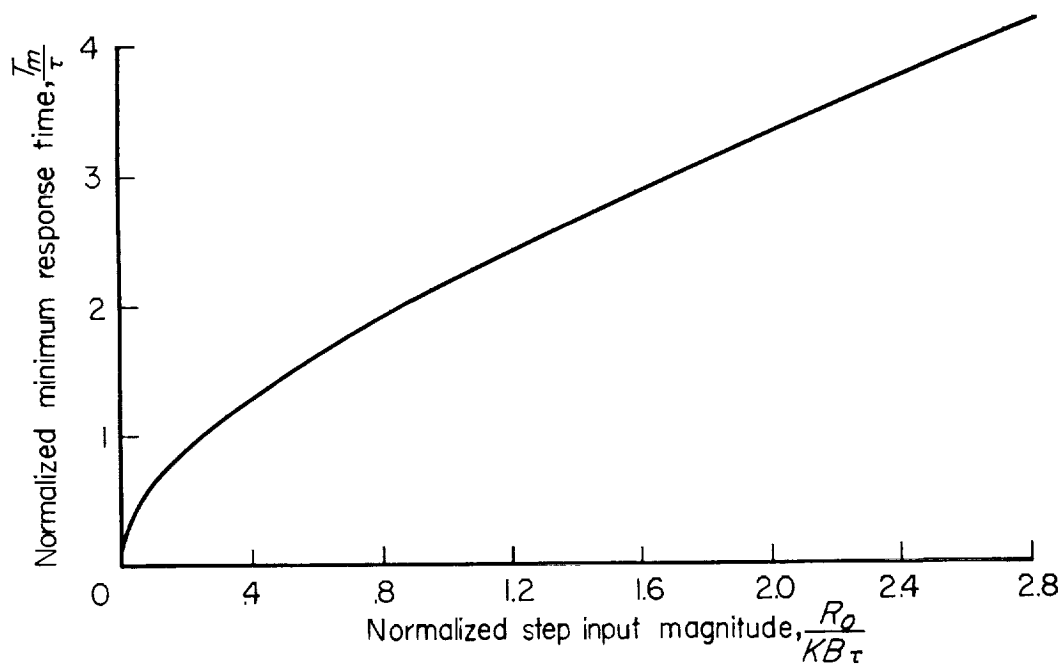
Case	Plant transfer function	References	Results or comments
1	$\frac{K}{s}$		$x(t) = B \operatorname{sgn} e(t)$
2	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)}$	11 for $\tau_1 = 0$ ; for $\tau_1/\tau_2 \geq 0.5$ optimum derived here	for $\tau_1 = 0$ $x(t) = B \operatorname{sgn} \left[ e(t) + \frac{\tau_2 \dot{e}(t)}{KB} \left[ \frac{1 - \ln \left( 1 + \frac{ \dot{e} }{KB} \right)}{ \dot{e} KB} \right] \right]$
3	$\frac{K}{s[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$	12	$x(t) = B \operatorname{sgn} f(e, \dot{e}, \ddot{e})$ complex function
4	$\frac{K}{s^2}$	11	$x(t) = B \operatorname{sgn} \left( e + \frac{\dot{e}}{2KB} \right)$
5	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)}$	13 for $\tau_1 = 0$	$x(t) = B \operatorname{sgn} f(e, \dot{e}, \ddot{e})$ for $\tau_1 = 0$
6	$\frac{K}{s^2[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$		$x(t) = B \operatorname{sgn} f(e, \dot{e}, \ddot{e}, \ddot{\ddot{e}})$
7	$\frac{K}{s^3}$	14	$x(t) = B \operatorname{sgn} f(e, \dot{e}, \ddot{e})$
8	$\frac{K}{s(\tau s - 1)}$	Approximate optimum derived here	
9	$\frac{K(\tau_1 s + 1)}{s[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$	None	
10	$\frac{K(\tau_1 s + 1)}{s(\tau_2 s + 1)(\tau_3 s - 1)}$		
11	$\frac{K(\tau_1 s + 1)}{s^2[(s^2/\omega_n^2) + (2\zeta/\omega_n)s + 1]}$	Approximate optimum derived here	
12	$\frac{K(\tau_1 s + 1)}{s^2(\tau_2 s + 1)(\tau_3 s - 1)}$		

Note: A number of the references use output derivative feedbacks rather than error derivatives. As has been pointed out, error derivatives are necessary if the optimum is to be obtained for arbitrary inputs.





(a) Optimum first reversal time.



(b) Minimum response time.

Figure 1.- Optimum response times for a type 1 second-order unstable plant.

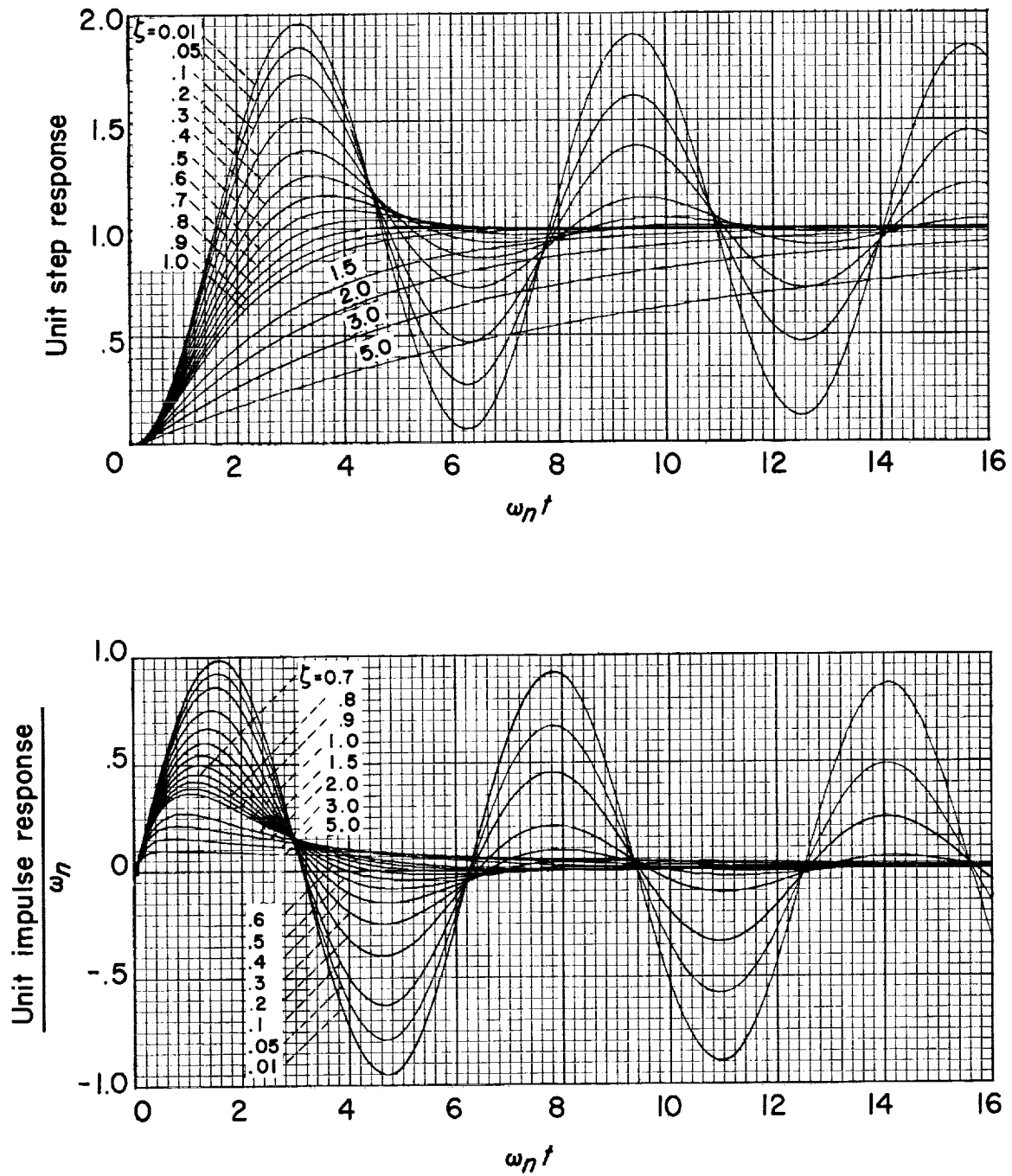
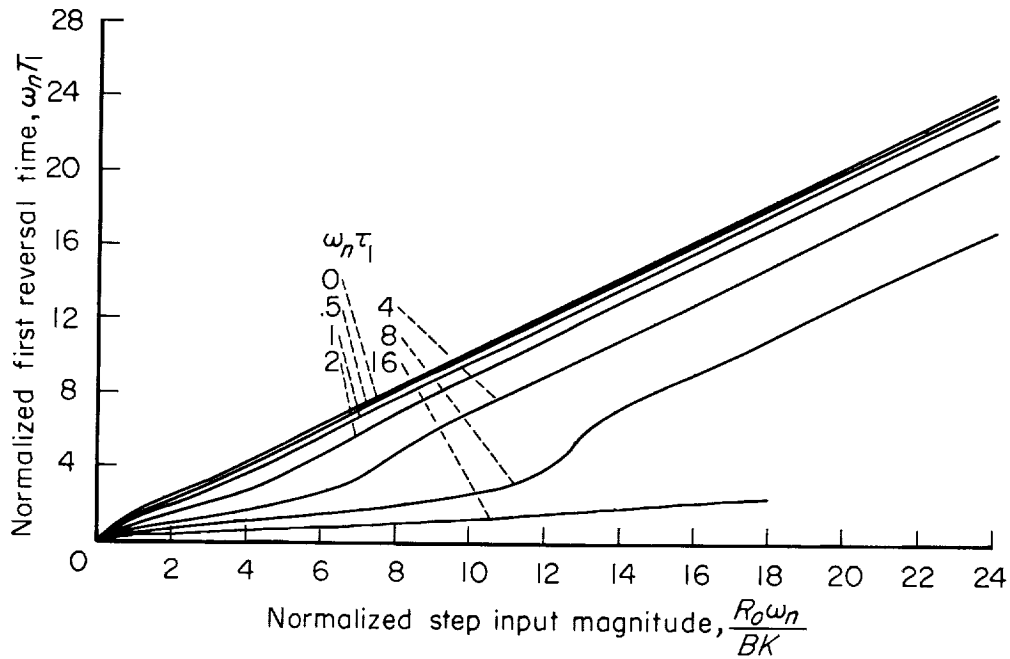
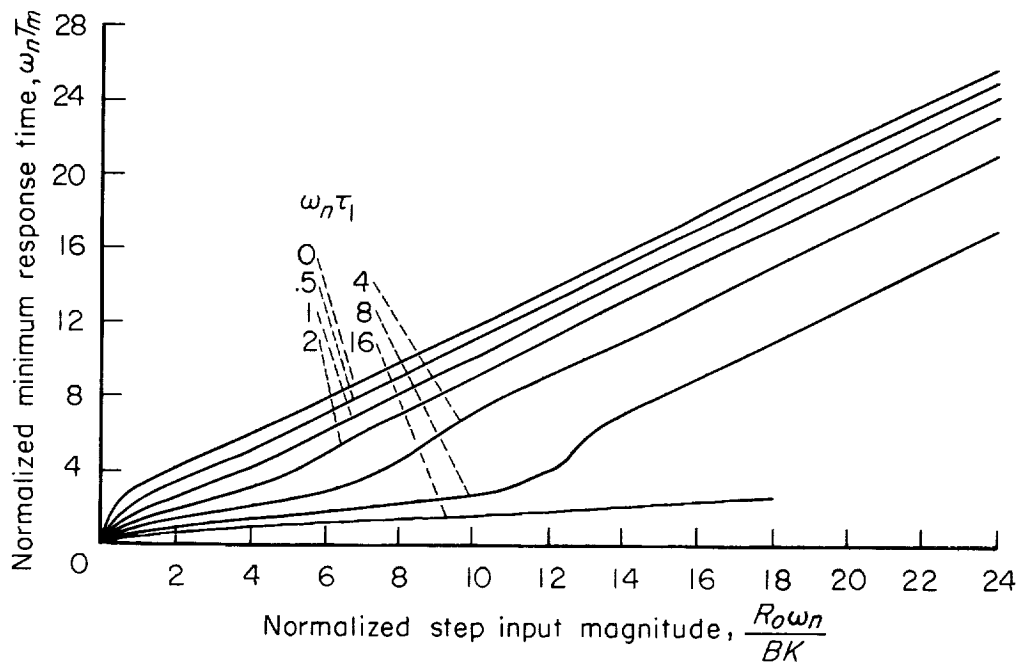


Figure 2.- Responses of a second-order system.

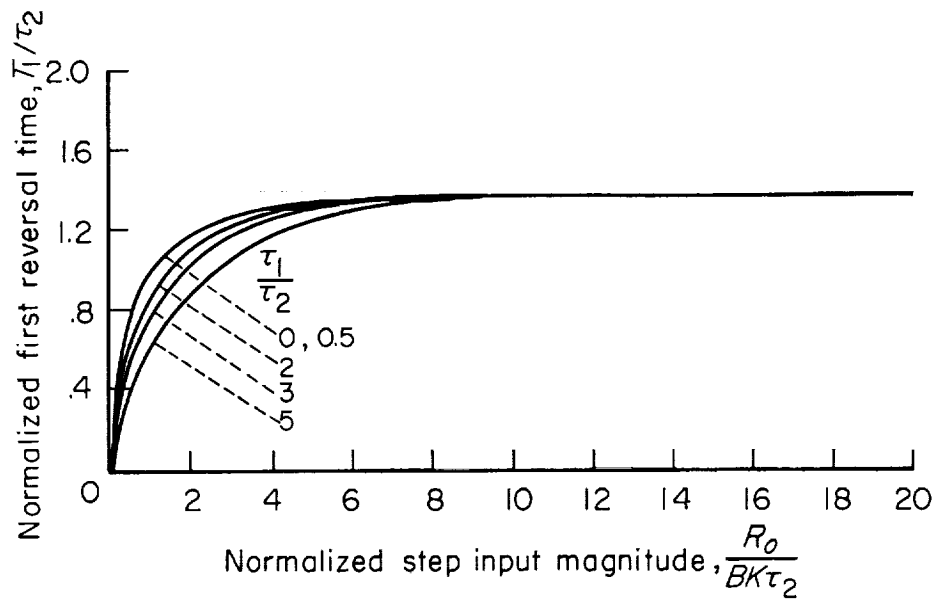


(a) Optimum first reversal time.

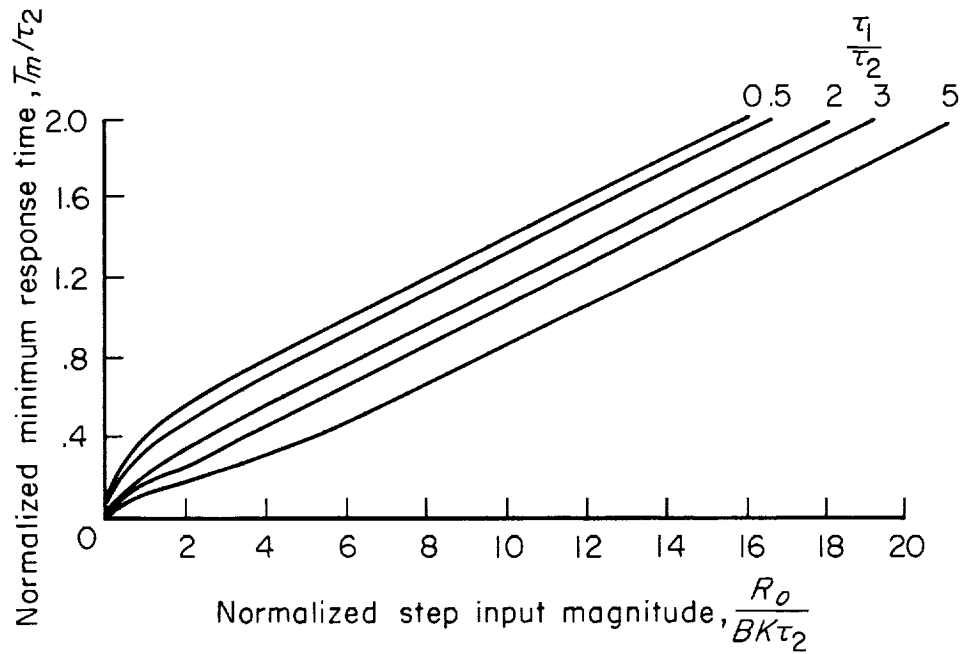


(b) Minimum response time.

Figure 3.- Optimum response times for a type 1 third-order plant with zero ( $\zeta = 0.5$ ).

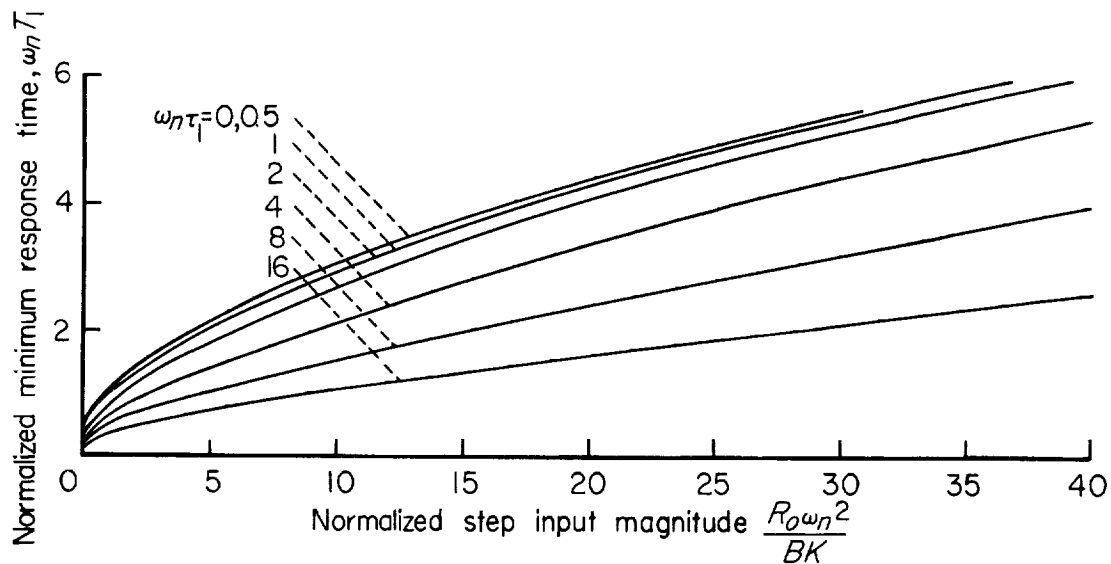


(a) Optimum first reversal time.

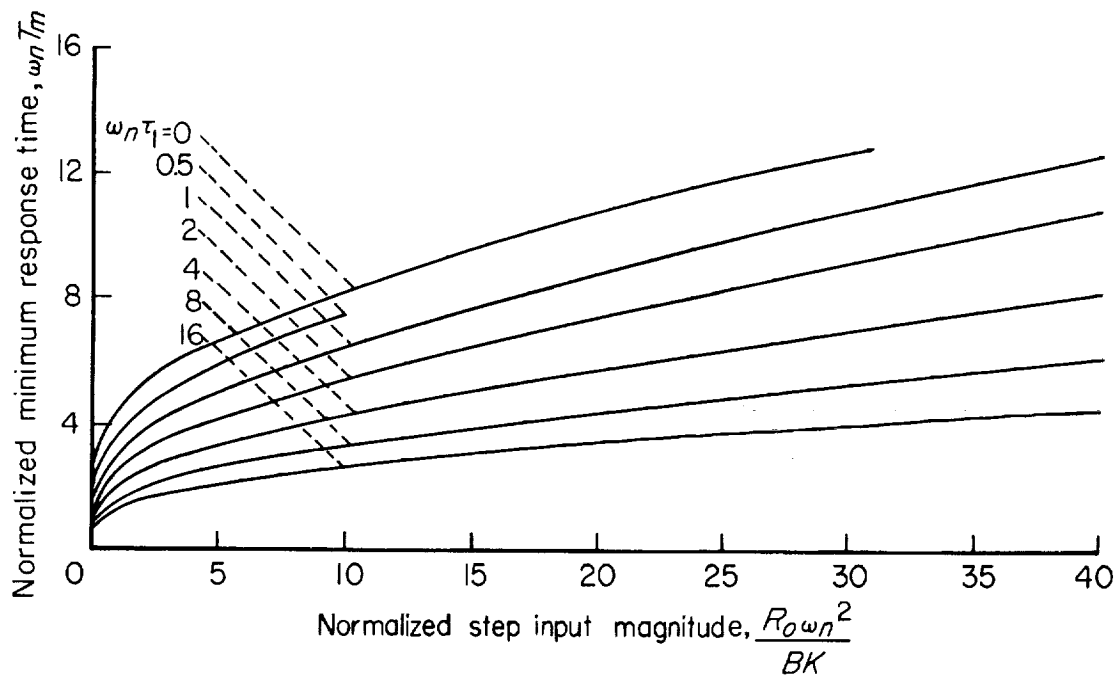


(b) Minimum response time.

Figure 4.- Optimum response times for a type 1 third-order unstable plant with zero ( $\tau_2/\tau_3 = 0.5$ ).

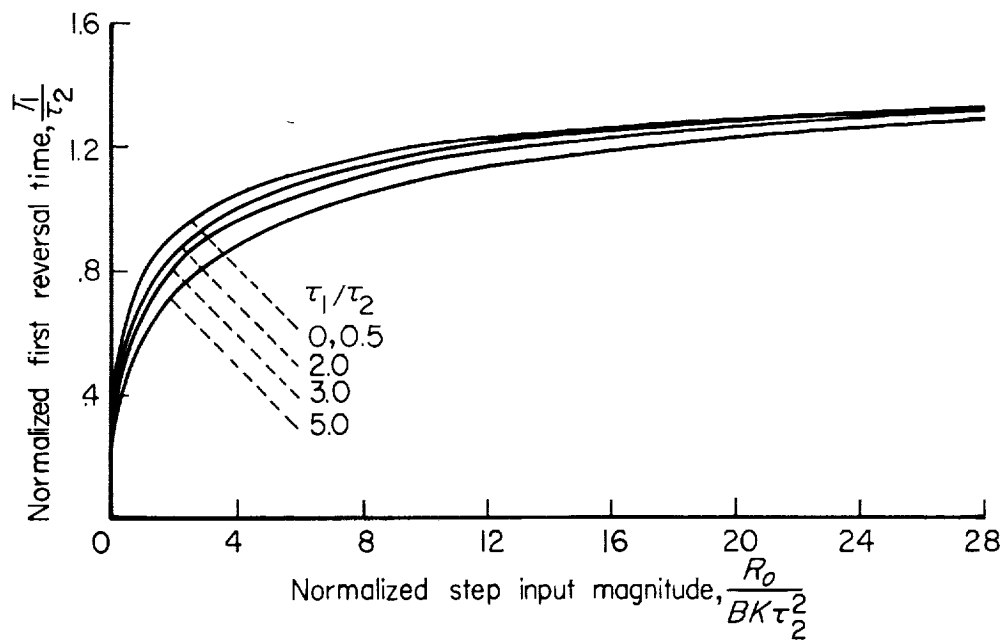


(a) Optimum first reversal time.

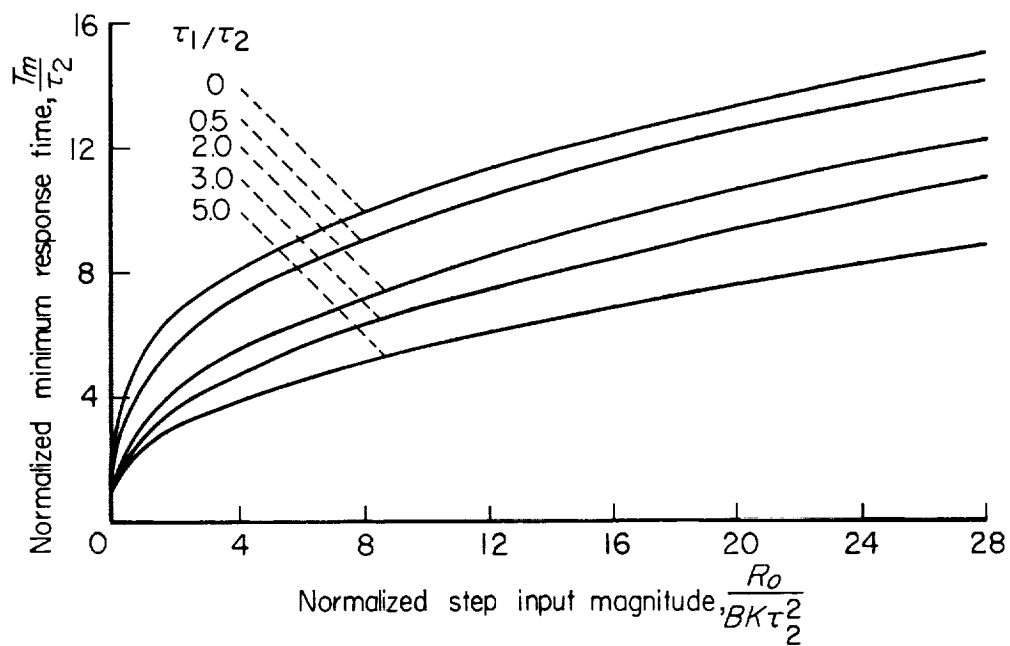


(b) Minimum response time.

Figure 5.- Optimum response times for a type 2 fourth-order plant with zero ( $\zeta = 0.5$ ).



(a) Optimum first reversal time.



(b) Minimum response time.

Figure 6.- Optimum response times for a type 2 fourth-order unstable plant with zero ( $\tau_2/\tau_3 = 0.5$ ).

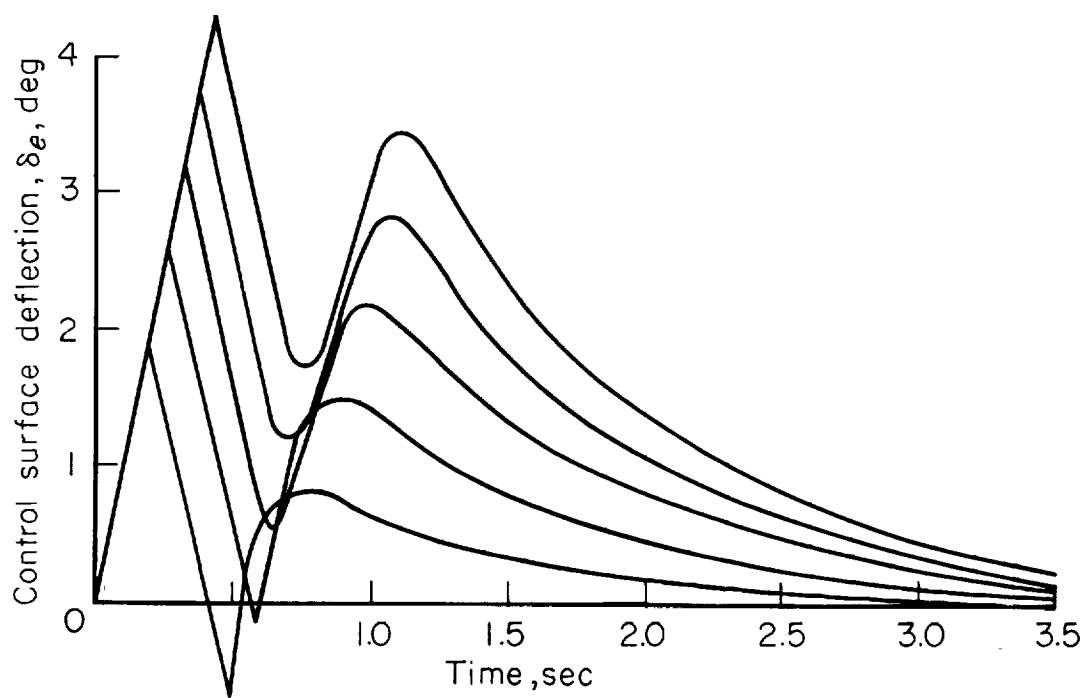
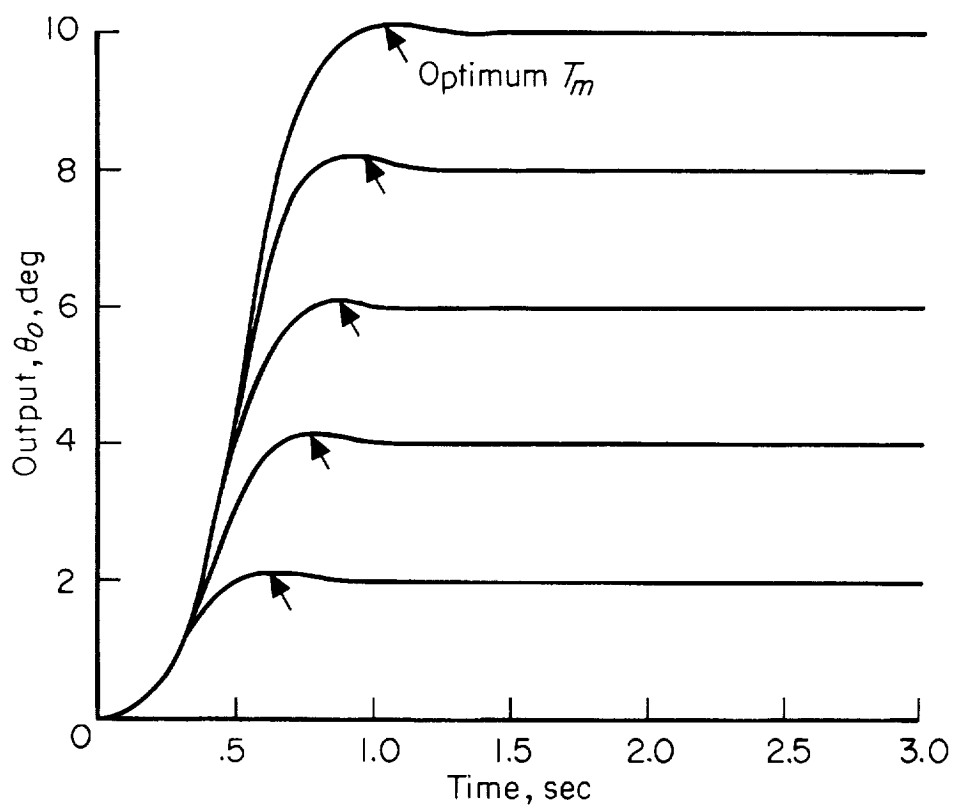


Figure 7.- Responses of attitude control system with stable aerodynamics.

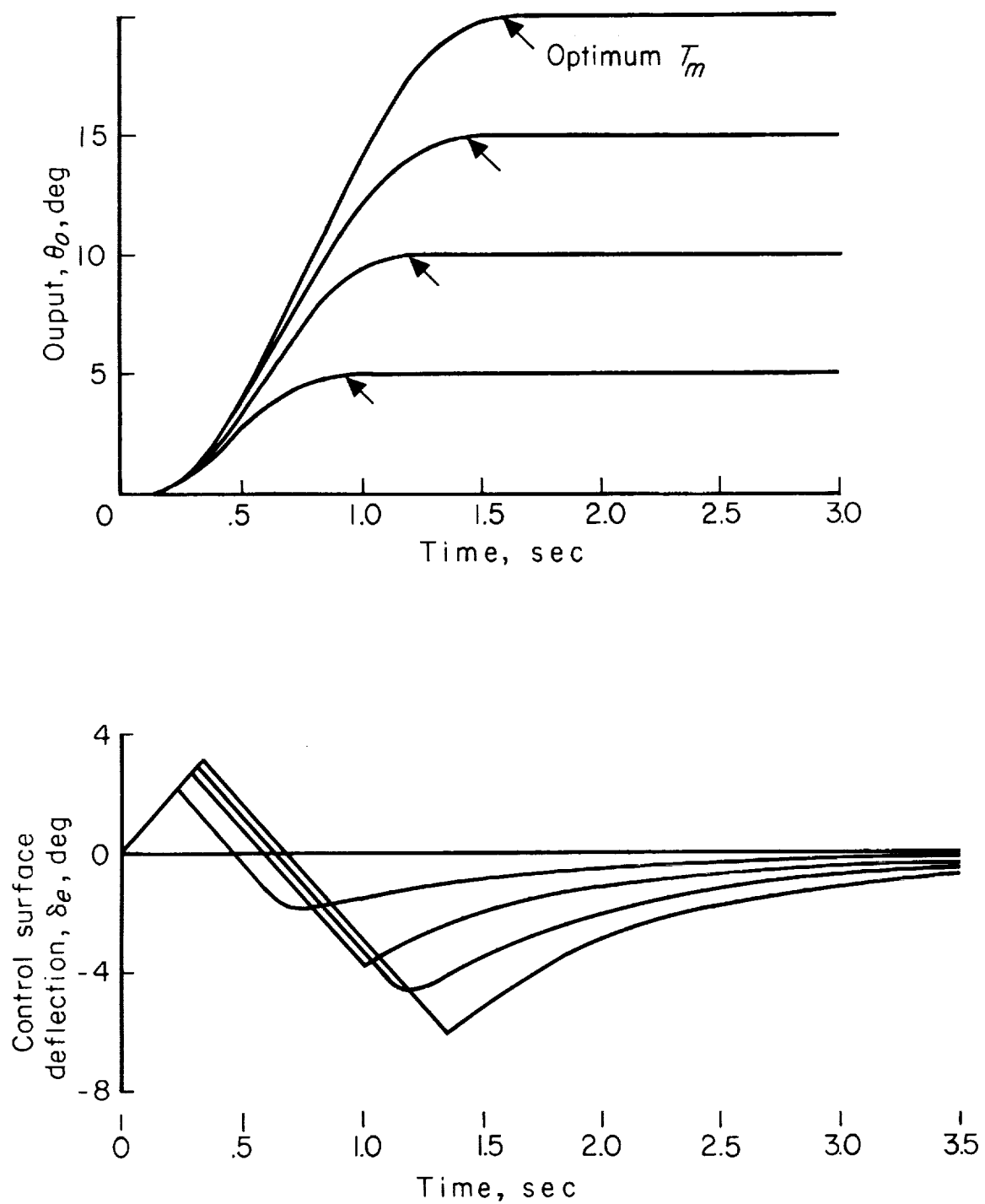
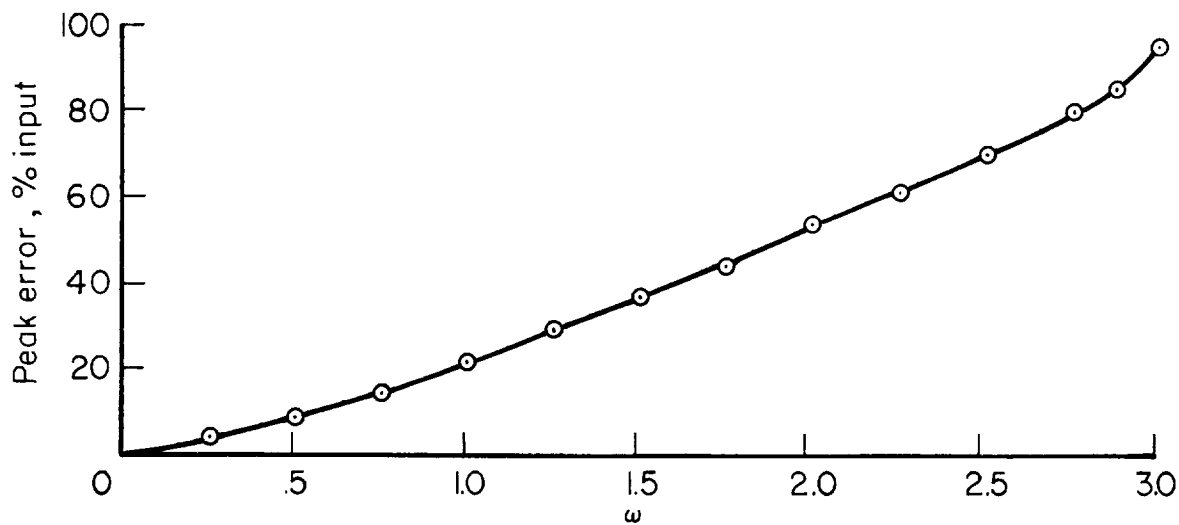
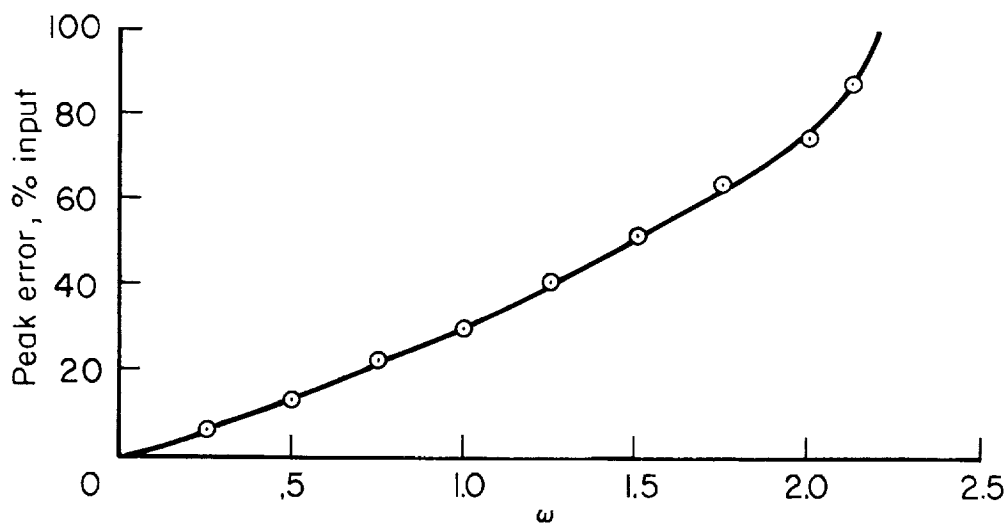


Figure 8.- Responses of attitude control system for unstable aerodynamics.



Input =  $10 \sin \omega t$ 

(a) Stable.



(b) Unstable.

Figure 9.- Error magnitude versus frequency for the two example attitude control systems.

